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香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八二年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1982

Pure Mathematics I

MARKING SCHEME

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Function

1. (a) For $0 < \lambda < 1$, let $\varphi(t) = \lambda t + (1 - \lambda) - t^\lambda$

$$\frac{d\varphi}{dt} = \lambda - \lambda t^{\lambda-1}$$

$$= \lambda \left(1 - \frac{1}{t^{1-\lambda}}\right) \begin{cases} = 0 & \text{if } t = 1 \\ < 0 & \text{if } 0 < t < 1 \\ > 0 & \text{if } t > 1 \end{cases}$$

$$\therefore \varphi(t) \geq \varphi(1) = 0 \quad \forall t > 0$$

i.e. $\lambda t + (1 - \lambda) \geq t^\lambda \quad \forall t > 0$.

Next, if either of α and β is zero, the second inequality is obvious.

If $\alpha, \beta > 0$, let $t = \frac{\alpha}{\beta}$

$$> 0$$

Substituting in the first inequality

$$\lambda \frac{\alpha}{\beta} + (1 - \lambda) \geq \left(\frac{\alpha}{\beta}\right)^\lambda$$

or $\lambda \alpha + (1 - \lambda) \beta \geq \alpha^\lambda \beta^{1-\lambda} \quad \forall \alpha, \beta \geq 0$.

(7 marks)

(b) Let $\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{q}$, then $0 < \lambda < 1$.

For each i , let $\alpha = a_i^p \geq 0$

$$\beta = b_i^q \geq 0$$

By (a) $\frac{1}{p} a_i^p + \frac{1}{q} b_i^q \geq (a_i^p)^{\frac{1}{p}} (b_i^q)^{\frac{1}{q}}$

$$= a_i b_i$$

$$\begin{aligned} \therefore \sum_{i=1}^n a_i b_i &\leq \sum_{i=1}^n \left(\frac{1}{p} a_i^p + \frac{1}{q} b_i^q \right) \\ &= \frac{1}{p} \sum_{i=1}^n a_i^p + \frac{1}{q} \sum_{i=1}^n b_i^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

(5 marks)

Question

1. (b) If all x_i or all y_i are zero, the inequality is obvious.

(Cont'd)

Otherwise, let $a_i = \frac{x_i}{\left[\sum_{r=1}^n x_r^p\right]^{\frac{1}{p}}} > 0$

$$\left[\sum_{r=1}^n x_r^p\right]^{\frac{1}{p}}$$

$b_i = \frac{y_i}{\left[\sum_{r=1}^n y_r^q\right]^{\frac{1}{q}}} > 0$

$$\left[\sum_{r=1}^n y_r^q\right]^{\frac{1}{q}}$$

Then $a_i, b_i \geq 0$ and $\sum_{i=1}^n a_i^p = \sum_{i=1}^n b_i^q = 1$.

By (b) $\sum_{i=1}^n \frac{x_i}{\left[\sum_{r=1}^n x_r^p\right]^{\frac{1}{p}}} \times \frac{y_i}{\left[\sum_{r=1}^n y_r^q\right]^{\frac{1}{q}}} \leq 1$

i.e. $\prod_{i=1}^n x_i y_i \leq \left[\prod_{i=1}^n x_i^p\right]^{\frac{1}{p}} \left[\prod_{i=1}^n y_i^q\right]^{\frac{1}{q}}$

(5 marks)

2. (a) Since

r_1, r_2, r_3, r_4 are roots of $f(x) = 0$,

Let $f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$

$$\frac{df}{dx} = (x - r_1) \frac{d}{dx} [(x - r_2)(x - r_3)(x - r_4)] + (x - r_2)(x - r_3)(x - r_4)$$

$$= (x - r_1) \left\{ (x - r_2) \frac{d}{dx} [(x - r_3)(x - r_4)] + (x - r_3)(x - r_4) \right\} + (x - r_2)(x - r_3)(x - r_4)$$

$$= (x - r_1)(x - r_2)(x - r_3) + (x - r_1)(x - r_2)(x - r_4) + (x - r_1)(x - r_3)(x - r_4) + (x - r_2)(x - r_3)(x - r_4)$$

$$= \sum_{i=1}^4 \frac{f(x)}{x - r_i} \quad \forall x \neq r_i (i = 1, 2, 3, 4)$$

(b) Since $f(r_i) = 0$ for $i = 1, 2, 3, 4$,

(4 marks)

$$\frac{f(x)}{x - r_i} = \frac{f(x) - f(r_i)}{x - r_i}$$

$$= \frac{(x^4 - r_i^4) + a_1(x^3 - r_i^3) + a_2(x^2 - r_i^2) + a_3(x - r_i)}{x - r_i}$$

$$= (x^3 + r_i x^2 + r_i^2 x + r_i^3) + a_1(x^2 + r_i x + r_i^2) + a_2(x + r_i) + a_3 \quad \forall x \neq r_i$$

(c) By (a) and (b), $f(x) = \sum \frac{f(x)}{x - r_i}$ (5 marks)

$$= \left[\sum_{i=1}^4 x^3 + x^2 \sum_{i=1}^4 r_i + x \sum_{i=1}^4 r_i^2 + \sum_{i=1}^4 r_i^3 \right] +$$

$$\left[\sum_{i=1}^4 a_1 x^2 + a_1 x \sum_{i=1}^4 r_i + a_1 \sum_{i=1}^4 r_i^2 \right] +$$

$$\left[\sum_{i=1}^4 a_2 x + a_2 \sum_{i=1}^4 r_i + \sum_{i=1}^4 a_3 \right]$$

$$= 4x^3 + (\sum r_i + \sum a_1)x^2 + (\sum r_i^2 + a_1 \sum r_i + \sum a_2)x + (\sum r_i^3 + a_1 \sum r_i^2 + a_2 \sum r_i + \sum a_3)$$

$$= 4x^3 + (S_1 + 4a_1)x^2 + (S_2 + a_1 S_1 + 4a_2)x + (S_3 + a_1 S_2 + a_2 S_1 + 4a_3)$$

(4 marks)

2 (d) Differentiating w.r.t. x, $f'(x) = 4x^3 + 3a_1x^2 + 2a_2x + a_3$

Comparing coefficients with $f'(x)$ in (c)

$$S_1 + 4a_1 = 3a_1$$

$$S_2 + a_1S_1 + 4a_2 = 2a_2$$

$$S_3 + a_1S_2 + a_2S_1 + 4a_3 = a_3$$

i.e.

$$S_1 + a_1 = 0$$

$$S_2 + a_1S_1 + 2a_2 = 0$$

$$S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0$$

$$b_1 = a_1$$

$$b_2 = a_1$$

$$b_3 = 2a_2$$

$$b_4 = a_1$$

$$b_5 = a_2$$

$$b_6 = 3a_3$$

Answer:

(4 marks)

3 (a) For $n = 1, 2, \dots$

$$x_{2n+1} = \frac{p^2 + x_{2n}}{1 + x_{2n}}$$

$$= \frac{p^2 + \frac{p^2 + x_{2n-1}}{1 + x_{2n-1}}}{1 + \frac{p^2 + x_{2n-1}}{1 + x_{2n-1}}}$$

$$= \frac{2p^2 + x_{2n-1} + p^2x_{2n-1}}{1 + p^2 + 2x_{2n-1}}$$

(3 marks)

$$x_{2n-1} - x_{2n+1} = \frac{x_{2n-1} + p^2x_{2n-1} + 2x_{2n-1}^2 - 2p^2 - x_{2n-1} - p^2x_{2n-1}}{1 + p^2 + 2x_{2n-1}}$$

$$= \frac{2(x_{2n-1}^2 - p^2)}{1 + p^2 + 2x_{2n-1}} \dots \dots \dots (*)$$

We shall next show inductively that $x_{2n-1} > p$ for $n = 1, 2, \dots$

Now

$$x_{2k+1} - p = \frac{2p^2 + x_{2k-1} + p^2x_{2k-1} - p - p^3 - 2px_{2k-1}}{1 + p^2 + 2x_{2k-1}}$$

$$= \frac{(x_{2k-1} - p)(p - 1)^2}{1 + p^2 + 2x_{2k-1}} \dots \dots \dots (**)$$

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If $k = 1, x_1 > p$.

If $x_{2k-1} > p$ for some $k \geq 1$,

$x_{2k+1} > p$ by (**)

1+1

$x_{2n-1} > p$ for $n = 1, 2, \dots$

and by (*) $x_{2n-1} > x_{2n+1}$ for $n = 1, 2, \dots$

(7 marks)

3 (b) Since $y_1 > y_2 > \dots > y_{n-1} > y_n > \dots > p$,

$\{y_n\}$ is monotonic decreasing and bounded.

$\therefore \{y_n\}$ converges.

Let $\lim_{n \rightarrow \infty} y_n = y$

$$\lim_{n \rightarrow \infty} x_{2n-1} = \lim_{n \rightarrow \infty} \frac{2p^2 + x_{2n-3} + p^2 x_{2n-3}}{1 + p^2 + 2x_{2n-3}}$$

$$= \frac{2p^2 + \lim_{n \rightarrow \infty} x_{2n-3} + p^2 \lim_{n \rightarrow \infty} x_{2n-3}}{1 + p^2 + 2 \lim_{n \rightarrow \infty} x_{2n-3}}$$

$$2y^2 + (1 + p^2)y = 2p^2 + y + p^2 y$$

$$y = p$$

Since $y_n > p > 1 \quad \forall n$,

$$\lim_{n \rightarrow \infty} y_n = p$$

(7 marks)

$$\begin{aligned} X_{2n+3} - X_{2n+1} &= \frac{2p^2 + (1+p^2)X_{2n+1}}{(1+p^2) + 2X_{2n+1}} - \frac{2p^2 + (1+p^2)X_{2n-1}}{(1+p^2) + 2X_{2n-1}} \\ &= \frac{(2p^2 + (1+p^2)X_{2n+1})(1+p^2 + 2X_{2n-1}) - (1+p^2 + 2X_{2n+1})(2p^2 + (1+p^2)X_{2n-1})}{((1+p^2) + 2X_{2n+1})((1+p^2) + 2X_{2n-1})} \\ &= \frac{(1+p^2)^2 (X_{2n+1} - X_{2n-1})}{((1+p^2) + 2X_{2n+1})(1+p^2 + 2X_{2n-1})} \end{aligned}$$

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(a) The probability that A gets k heads

$$= C_k^m \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{m-k}$$

$$= C_k^m \left(\frac{1}{2}\right)^m$$

\therefore the probability that A and B both get k heads

$$= C_k^m \left(\frac{1}{2}\right)^m C_k^n \left(\frac{1}{2}\right)^n = C_k^m C_k^n \left(\frac{1}{2}\right)^{m+n} \quad k \leq n$$

$$\text{Required probability} = \sum_{k=0}^n C_k^m C_k^n \left(\frac{1}{2}\right)^{m+n}$$

$$\text{But } (1+x)^{m+n} = \sum_{i=0}^{m+n} C_i^{m+n} x^i$$

$$(1+x)^m (1+x)^n = \left(\sum_{k=0}^m C_k^m x^k\right) \left(\sum_{k=0}^n C_k^n x^k\right)$$

Since $(1+x)^{m+n} = (1+x)^m (1+x)^n$, by comparing coefficient of x^n ,

$$C_n^{m+n} = \sum_{k=0}^m C_k^m C_{n-k}^n$$

$$= \sum_{k=0}^n C_k^m C_k^n$$

(8 marks)

(b) Let x_n be the probability that A wins,

y_n be the probability that A and B draw.

$$\frac{y_{n+1}}{y_n} = \frac{C_{n+1}^{2n+2} \left(\frac{1}{2}\right)^{2n+2}}{C_n^{2n} \left(\frac{1}{2}\right)^{2n}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \left(\frac{1}{2}\right)^2$$

$$= \frac{2n+1}{2n+2} < 1$$

$$\therefore y_{n+1} < y_n$$

By symmetry, the probability that B wins = probability that A wins.

$$2x_n + y_n = 1$$

Since $y_{n+1} < y_n$

$$x_{n+1} > x_n$$

$$\text{For } n=3, y_3 = \frac{5}{16}$$

$$x_3 = \frac{11}{32} > y_3$$

5. (a) For any $A, B \subset X$,

$$A, B \subset A \cup B \implies f[A] \cup f[B] \subset f[A \cup B]$$

$$\text{and } f[A] \cup f[B] \subset f[A \cup B].$$

Next, for any $y, y \in f[A \cup B] \implies \exists x \in A \cup B$ s.t. $f(x) = y$
 $\implies (\exists x \in A \text{ s.t. } f(x) = y)$ or
 $(\exists x \in B \text{ s.t. } f(x) = y)$

$$\therefore y \in f[A] \cup f[B]$$

$$\text{and } f[A \cup B] \subset f[A] \cup f[B]$$

$$\text{Hence } f[A \cup B] = f[A] \cup f[B] \quad \forall A, B \subset X.$$

(5 marks)

(b) Suppose $f[A \cap B] = f[A] \cap f[B] \quad \forall A, B \subset X$.

Let $A = \{x_1\}$ and $B = \{x_2\}$ be any two singletons in X .

Since $f[A] \cap f[B] = f[A \cap B]$, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

$$\text{i.e. } f(x_1) = f(x_2) \implies x_1 = x_2$$

$\therefore f$ is injective.

(5 marks)

(c) Suppose f is bijective.

For any $A \subset X$, $y \in f[X \setminus A] \implies \exists x \in X \setminus A$ s.t. $y = f(x)$

$$\implies y \notin f[A] \text{ (since } f \text{ is injective)}$$

$$\implies y \in Y \setminus f[A].$$

$$\therefore f[X \setminus A] \subset Y \setminus f[A].$$

For any $y, y \in Y \setminus f[A] \implies \exists x \in X$ s.t. $f(x) = y$ (since f is surjective)

Obviously $x \notin A$, $\therefore x \in X \setminus A$

$$\text{i.e. } y \in f[X \setminus A]$$

$$\text{and } Y \setminus f[A] \subset f[X \setminus A].$$

$$\text{Hence } f[X \setminus A] = Y \setminus f[A] \quad \forall A \subset X.$$

(7 marks)

6. (a) Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} t & t \\ t & t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

$$\Leftrightarrow \begin{cases} t(x+y) = 2tx \\ t(x+y) = 2ty \\ t(z+w) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \\ z = -w \end{cases} \quad \forall t \neq 0$$

$$\therefore X = \begin{pmatrix} r & r \\ s & -s \end{pmatrix}, \quad r, s \in \mathbb{R}.$$

Let $r = s = 1$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $|Q| = -2 \neq 0$.

$$Q \text{ is non-singular and } \begin{pmatrix} t & t \\ t & t \end{pmatrix} = Q^{-1} \begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix} Q.$$

(4 marks)

(b) Consider the mapping $f: G \rightarrow H$ such that

$$f\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) = Q^{-1} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} Q.$$

$$= \begin{pmatrix} \frac{t}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{t}{2} \end{pmatrix} \text{ for all } \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in G.$$

Clearly f is bijective.

$$\begin{aligned} \text{Further, for all } A, B \in G, \quad f(A)f(B) &= (Q^{-1}AQ)(Q^{-1}BQ) \\ &= Q^{-1}ABQ \\ &= f(AB) \end{aligned}$$

Let E be a multiplicative identity in G , i.e.

$$EA = AE = A \quad \forall A \in G.$$

Since f is bijective, every matrix in H can be written as $f(A)$ for exactly one A in G . Then for any matrix $f(A)$ in H ,

$$\begin{aligned} f(E)f(A) &= f(EA) = f(A) \\ f(A)f(E) &= f(AE) = f(A) \end{aligned}$$

$\therefore f(E)$ is a multiplicative identity in H .

(7 marks)

6 (c) Multiplication is closed in G since $\begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t_1 t_2 & 0 \\ 0 & 0 \end{pmatrix} \in G$.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity and $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 0 \end{pmatrix}$ is the inverse of $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$.

$t \neq 0$. $\therefore G$ is a group.

Further, for any $f(A), f(B)$ in H , $f(A)f(B) = f(AB) \in H$ since $AB \in G$

\therefore multiplication is closed in H .

For any $f(A)$ in H , $A^{-1} \in G$ and $f(A)f(A^{-1}) = f(AA^{-1}) = f(E)$

$f(A^{-1})f(A) = f(A^{-1}A) = f(E)$

$\therefore f(A)$ has an inverse.

Together with (b), H is a group.

(6 marks)

7 (a) $Ax = \lambda x \Leftrightarrow Ax = \lambda Ix$
 $\Leftrightarrow (A - \lambda I)x = 0$

This equation has a non-zero solution

iff $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda + 2 = 0$
 $\lambda = 1$ or 2 .

Let $\lambda_1 = 1, \lambda_2 = 2$.

(3 marks)

(b) Let $Ax_1 = x_1$
 $Ax_2 = 2x_2$

Then $Ax = \begin{pmatrix} x_{11} & 2x_{12} \\ x_{21} & 2x_{22} \end{pmatrix}$

But $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 3x_{11} - x_{21} & 3x_{12} - x_{22} \\ 2x_{11} & 2x_{12} \end{pmatrix}$

$\begin{cases} 2x_{11} - x_{21} = 0 \\ x_{12} - x_{22} = 0 \end{cases}$

$|X| = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

$= x_{11}x_{22} - x_{12}x_{21}$
 $= x_{11}x_{22} - 2x_{11}x_{22}$
 $= -x_{11}x_{22}$

Now $|X| = 0 \Rightarrow x_{11} = 0$ or $x_{22} = 0$
 $\Rightarrow x_{21} = 0$ or $x_{12} = 0$
 $\Rightarrow x_1 = 0$ or $x_2 = 0$, which is not true.

$\therefore X$ is non-singular

7 marks

7. (c) (i) $A \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \lambda_1 \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} \\ \lambda_1 x_{21} \end{pmatrix}$
 $A \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \lambda_2 \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

$$\therefore A \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} \\ \lambda_1 x_{21} & \lambda_2 x_{22} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A = X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1}$$

$$A^n = \left[X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1} \right] \left[X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1} \right] \dots \left[X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1} \right]$$

$$= X \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} X^{-1}$$

(ii) By (a), let $\lambda_1 = 1$, $\begin{cases} 2x_1 - x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution of $Ax = 1x$.

Let $\lambda_2 = 2$, $\begin{cases} x_1 - x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of $Ax = 2x$.

$$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2^{n+1} - 1 & 1 - 2^n \\ 2^{n+1} - 2 & 2 - 2^n \end{pmatrix}$$

(7 marks)

$$u + v + 1 = 0 \Rightarrow \frac{u}{u} + \frac{v}{v} + 1 = 0$$

$$\Rightarrow \frac{u}{u} + \frac{v}{v} + 1 = 0$$

$$\therefore \frac{1}{u} + \frac{1}{v} + 1 = 0$$

If $\frac{1}{u} + \frac{1}{v} + 1 = 0$, $\frac{u+v+uv}{uv} = 0$ (3 marks)

$$\Rightarrow |u| = |v| = 1$$

$$\begin{aligned} u + v + uv &= 0 \\ u &= -v(1+u) \\ u &= -v(-v) \\ &= v^2 \end{aligned}$$

(since $1+u+v=0$)

$$\begin{aligned} v^3 &= uv \\ &= -(u+v) \\ &= +1 \end{aligned}$$

$\Rightarrow |v| = 1$
and hence $|u| = 1$

$|u| = |v| = 1$ iff $\frac{1}{u} + \frac{1}{v} + 1 = 0$ (3 marks)

Next, $u + v + 1 = 0 \Rightarrow u^2 + v^2 + 1 + 2(u+v+uv) = 0$

If $\frac{1}{u} + \frac{1}{v} + 1 = 0$, $\frac{u+v+uv}{uv} = 0 \Rightarrow u^2 + v^2 + 1 = 0$

If $u^2 + v^2 + 1 = 0$, $u + v + uv = 0 \Rightarrow \frac{1}{u} + \frac{1}{v} + 1 = 0$

$u^2 + v^2 + 1 = 0 \Leftrightarrow \frac{1}{u} + \frac{1}{v} + 1 = 0 \Leftrightarrow |u| = |v| = 1$ (4 marks)

(b) The three sides AB, BC and CA are, respectively,
 $|z_2 - z_1|$, $|z_3 - z_2|$ and $|z_1 - z_3|$.

Consider the two complex numbers $u = \frac{z_2 - z_1}{z_1 - z_3}$, $v = \frac{z_3 - z_2}{z_1 - z_3}$

$$u + v + 1 = \frac{z_2 - z_1}{z_1 - z_3} + \frac{z_3 - z_2}{z_1 - z_3} + 1$$

$$= 0$$

$$\left| \frac{z_2 - z_1}{z_1 - z_3} \right| = \left| \frac{z_3 - z_2}{z_1 - z_3} \right| = 1$$

$$\Leftrightarrow \left(\frac{z_2 - z_1}{z_1 - z_3} \right)^2 + \left(\frac{z_3 - z_2}{z_1 - z_3} \right)^2 + 1 = 0$$

$$\Leftrightarrow (z_2 - z_1)^2 + (z_3 - z_2)^2 + (z_1 - z_3)^2 = 0$$

$$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$$

i.e. $|z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$

iff $z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$

(7 marks)

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1. (a) Putting $y = mx + c$ in (E).

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\frac{a^2 m^2 + b^2}{a^2 b^2} x^2 + \frac{2mc}{b^2} x + \frac{c^2 - b^2}{b^2} = 0$$

Condition for tangency is

$$\frac{4m^2 c^2}{b^4} - 4 \left(\frac{a^2 m^2 + b^2}{a^2 b^2} \right) \left(\frac{c^2 - b^2}{b^2} \right) = 0$$

$$\text{i.e. } a^2 m^2 + b^2 - c^2 = 0.$$

Since $P(h, k)$ lies on the tangent,

$$c = k - mh.$$

$$a^2 m^2 + b^2 - (k - mh)^2 = 0$$

$$\text{or } (a^2 - h^2)m^2 + 2hkm + (b^2 - k^2) = 0 \quad \dots \dots \dots (1) \quad (6 \text{ marks})$$

(b) The tangents of the angles between tangents from P to the line $y = nx$ are given by

$$\frac{m_1 - n}{1 + m_1 n}$$

$$\frac{m_2 - n}{1 + m_2 n}$$

$$\text{If these angles are equal } \frac{m_1 - n}{1 + m_1 n} = \frac{n - m_2}{1 + m_2 n}$$

$$\text{Expanding } (1 - n^2)(m_1 + m_2) + 2n(m_1 m_2 - 1) = 0 \quad \dots \dots \dots (2) \quad (5 \text{ marks})$$

$$(c) \text{ From (1), } m_1 + m_2 = \frac{-2hk}{a^2 - h^2}$$

$$m_1 m_2 = \frac{b^2 - k^2}{a^2 - h^2}$$

$$\text{Sub. in (2), } (1 - n^2) \left(\frac{-2hk}{a^2 - h^2} \right) + 2n \left(\frac{b^2 - k^2}{a^2 - h^2} - 1 \right) = 0$$

$$-2hk(1 - n^2) + 2n(b^2 - a^2 + h^2 - k^2) = 0$$

$$2nh^2 - 2nk^2 - 2(1 - n^2)hk + 2n(b^2 - a^2) = 0$$

\therefore equation of locus of P is

$$nx^2 - ny^2 + (n^2 - 1)xy - n(a^2 - b^2) = 0.$$

(6 marks)

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$$\cos(n+1)\theta = \cos n\theta \cos\theta - \sin n\theta \sin\theta$$

$$\begin{aligned} &= 2 \cos(n-1)\theta \cos\theta - \sin(n-1)\theta \sin\theta + \cos(n-1)\theta \sin\theta \\ &= 2 \cos(n-1)\theta \cos\theta - \sin(n-1)\theta \sin\theta - \cos(n-1)\theta \sin\theta \\ &= 2 \cos(n-1)\theta \cos\theta - \sin(n-1)\theta \cos\theta \sin\theta - \cos(n-1)\theta \sin\theta \\ &= 2 \cos(n-1)\theta \cos\theta - \cos(n-1)\theta (\sin\theta \sin\theta + \cos\theta \sin\theta) \\ &= 2 \cos(n-1)\theta \cos\theta - \cos(n-1)\theta (\sin\theta + \cos\theta \sin\theta) \\ &= 2 \cos(n-1)\theta \cos\theta - \cos(n-1)\theta (1 + \cos\theta \sin\theta) \\ &= 2 \cos(n-1)\theta \cos\theta - \cos(n-1)\theta \end{aligned}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (4 \text{ marks})$$

from def $\rightarrow T_0(x) = 1$ and $T_1(x) = x$ ← from def

from above $\rightarrow T_2(x) = 2x^2 - 1$ is a polynomial of degree 2 with leading coefficient 2^1 .

Since $T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x)$, if, for $1 \leq n \leq k$, $T_{n+1}(x)$ is a polynomial of degree $k+1$ with leading coefficient 2^k , then $T_{k+2}(x)$ is a polynomial of degree $k+2$ with leading coefficient 2^{k+1} .

$\therefore T_n(x)$ is a polynomial of degree n with leading coefficient $2^{n-1} \forall n \geq 1$.

(4 marks)

$$(b) \quad \cos\theta = \frac{1}{2} [(\cos\theta + i \sin\theta) + (\cos\theta - i \sin\theta)] \quad (1 = -\frac{1}{2})$$

$$\begin{aligned} \therefore \cos^n\theta &= \frac{1}{2^n} \sum_{k=0}^n C_k^n (\cos\theta)^k (\cos\theta - i \sin\theta)^{n-k} \\ &= \frac{1}{2^n} \sum_{k=0}^n C_k^n (\cos k\theta) (\cos(k-n)\theta) \\ &= \frac{1}{2^n} \sum_{k=0}^n C_k^n \cos(2k-n)\theta \\ &= \frac{1}{2^n} \sum_{k=0}^n C_k^n \cos(2k-n)\theta \quad (\text{since imaginary part} = 0) \\ &= \frac{1}{2^n} \sum_{k=0}^n C_k^n \cos(n-2k)\theta \end{aligned}$$

$$\therefore a_k = \frac{1}{2^n} C_k^n$$

(6 marks)

2. (c) Since $\sum_{k=0}^n C_k^n \cos(n-2k)\theta = C_{n-k}^n \cos(2k-n)$, (by putting $\theta = \cos^{-1}x$)

by (b), $x^n = \frac{1}{2^n} \sum_{k=0}^n C_k^n \cos(n-2k)\theta \cos^n x$

$$= \frac{1}{2^n} \sum_{k=0}^n 2C_k^n \cos(n-2k)\theta \cos^n x$$

$$= \frac{1}{2^{n-1}} \sum_{k=0}^n C_k^n T_{n-2k}(x), \quad n = 1, 3, 5, \dots$$

3 marks

put $j = n-2k, k = n-j$

$$k = \frac{n+j}{2}, \quad j = \frac{n-k}{2}$$

$$k = n, \quad j = 0$$

$$\sum_{j=0}^n C_{n-j}^n \cos[(n-2(n-j))\theta \cos^{-1}x]$$

$$= \sum_{j=0}^n C_{n-j}^n \cos(n-2(n-j))\theta \cos^{-1}x$$

$$= \sum_{j=0}^n C_{n-j}^n \cos(n-2(n-j))\theta \cos^{-1}x$$

$$= \sum_{j=0}^n C_{n-j}^n \cos(n-2(n-j))\theta \cos^{-1}x$$

$$x^n = \frac{1}{2^n} \sum_{j=0}^n C_{n-j}^n \cos(n-2(n-j))\theta \cos^n x$$

$$= \frac{1}{2^{n-1}} \sum_{j=0}^n C_{n-j}^n \cos(n-2(n-j))\theta \cos^n x$$

$$= \frac{1}{2^{n-1}} \sum_{j=0}^n C_{n-j}^n T_{n-2(n-j)}(x)$$

Since $\lim_{h \rightarrow 0} f(x_0) = f(x_0)$ and $f(x_0 + h) = f(x_0) + f(h)$,

$$\lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} f(x_0 + h) - \lim_{h \rightarrow 0} f(x_0) = 0$$

$$f(x) = \lim_{h \rightarrow 0} f(x) + \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} [f(x) + f(h)]$$

$$= \lim_{h \rightarrow 0} f(x+h) \quad \forall x \in \mathbb{R}$$

i.e. f is cont. at every $x \in \mathbb{R}$.

(5 marks)

(b) We shall first induce on n for $n \geq 0$.

$$\begin{aligned} \text{For } n=0, \quad f(0) &= f(0+0) \\ &= f(0) + f(0) \\ f(0) &= 0 \end{aligned}$$

Assume $f(kx) = kf(x)$ for some $k \geq 0$.

$$\begin{aligned} \text{Then } f((k+1)x) &= f(kx) + f(x) \\ &= kf(x) + f(x) \\ &= (k+1)f(x) \end{aligned}$$

$$\text{Hence } f(nx) = nf(x) \quad \forall n \geq 0$$

$$\begin{aligned} \text{Next } f(x) + f(-x) &= f(x-x) = 0 \quad \forall x \\ \Rightarrow f(-x) &= -f(x) \end{aligned}$$

$$\therefore \text{ if } n < 0, \quad f(nx) = -nf(-x) = nf(x)$$

Let $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}, q \neq 0$.

$$\begin{aligned} \text{By (b), } \quad qf\left(\frac{p}{q}\right) &= f\left(q \frac{p}{q}\right) \\ &= f(p) \\ &= pf(1) \\ \therefore f(r) &= f(1) = r \end{aligned}$$

of rational numbers

(8 marks)

(c) For any $x \in \mathbb{R}$, let $\{a_n\}$ be a sequence such that $\lim_{n \rightarrow \infty} a_n = x$.

As f is continuous,

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} f(a_n) \\ &= \lim_{n \rightarrow \infty} f(1) \cdot a_n \\ &= f(1) \lim_{n \rightarrow \infty} a_n \\ &= f(1) \cdot x \end{aligned}$$

The answer follows where $k = f(1)$.

(a) For $n=0$,

$$\begin{aligned} \text{R.S.} &= I_0 = \int_0^1 e^t dt \\ &= e^t \Big|_0^1 = e - 1 \end{aligned}$$

$$\text{L.S.} = (-1) + e = \text{R.S.}$$

Assume that $I_k = (-1)^{k+1} k! + e \sum_{i=0}^k (-1)^i \frac{k!}{(k-i)!}$ for some $k \geq 0$,

$$I_{k+1} = \int_0^1 e^t t^{k+1} dt$$

$$\text{Put } u = t^{k+1}, \quad dv = e^t dt$$

$$du = (k+1)t^k, \quad v = e^t$$

$$\begin{aligned} I_{k+1} &= e^t t^{k+1} \Big|_0^1 - (k+1) \int_0^1 t^k e^t dt \\ &= e - (k+1) I_k \\ &= e - [(-1)^{k+1} k! - e \sum_{i=0}^k (-1)^i \frac{k!}{(k-i)!}] (k+1) \\ &= e + (-1)^{k+2} (k+1)! - e (k+1) \sum_{i=0}^k (-1)^i \frac{k!}{(k-i)!} \\ &= (-1)^{k+2} (k+1)! - e \sum_{i=0}^k (-1)^i \frac{(k+1)!}{(k-i)!} + e \\ &= (-1)^{k+2} (k+1)! + e \sum_{j=1}^{k+1} (-1)^j \frac{(k+1)!}{(k+1-j)!} + e, \text{ where } j = k-i \\ &= (-1)^{k+2} (k+1)! + e \sum_{j=0}^{k+1} (-1)^j \frac{(k+1)!}{(k+1-j)!} \end{aligned}$$

$$\therefore I_n = (-1)^{n+1} n! + e \sum_{i=0}^n (-1)^i \frac{n!}{(n-i)!} \quad \forall n \geq 0$$

6 marks

(b) For $0 \leq t \leq 1, n \geq 1, t^n \leq e^t t^n \leq e t^n < e^{t+1}$

$$\begin{aligned} \therefore I_n &= \int_0^1 e^t t^n dt \leq \int_0^1 e t^n dt \\ &= e \frac{t^{n+1}}{n+1} \Big|_0^1 \\ &= \frac{e}{n+1} \\ &< \frac{e}{n} \end{aligned}$$

4. (b) Also $I_n \geq \int_0^1 t^n dt$

$$= \frac{t^{n+1}}{n+1} \Big|_0^1$$

$$= \frac{1}{n+1}$$

i.e. $\frac{1}{n+1} \leq I_n \leq \int_0^1 e^{nt} dt < \frac{e}{n}$

4 marks

(c) By (a) and (b) $\frac{1}{n+1} \leq (-1)^{n+1} n! + e \sum_{i=0}^n \frac{(-1)^i n!}{(n-i)!} < \frac{e}{n}$

Assume, for contradiction, that $e = \frac{p}{q}$, where $p, q \geq 1$.

Then $\frac{1}{n+1} \leq (-1)^{n+1} n! + \frac{p}{q} \sum_{i=0}^n \frac{(-1)^i n!}{(n-i)!} < \frac{p}{qn} \quad \forall n \geq 1$

$\frac{q}{n+1} \leq (-1)^{n+1} n! q + p \sum_{i=0}^n \frac{(-1)^i n!}{(n-i)!} < \frac{p}{n} \quad \forall n \geq 1$

Since $(-1)^{n+1} n! q + p \sum_{i=0}^n \frac{(-1)^i n!}{(n-i)!}$ is an integer

and $\frac{q}{n+1} > 0$,

taking n_0 to be greater than p , $n_0 > p \implies \frac{q}{n_0} < 1$

then $0 < \frac{q}{n_0+1} \leq (-1)^{n_0+1} n_0! q + p \sum_{i=0}^{n_0} \frac{(-1)^i n_0!}{(n_0-i)!} < \frac{p}{n_0} < 1$,

which is a contradiction.

$\therefore e$ cannot be a rational number.

7 marks

5. (a) Suppose $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exists.

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{1}{2} \left[\frac{f(x+h) - f(x)}{h} + \frac{f(x-h) - f(x)}{-h} \right]$$

$$\lim_{h \rightarrow 0} \frac{1}{2} \left[\frac{f(x+h) - f(x)}{h} + \frac{f(x-h) - f(x)}{-h} \right]$$

$$= \frac{1}{2} \left[\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \right]$$

$$= f'(x)$$

$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$ exists.

(4 marks)

(b) Put $x = 0$.

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x-h)}{2h} = \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h} + h \sin \frac{1}{-h}}{2h}$$

$$= \lim_{h \rightarrow 0} \frac{0}{2h} = 0$$

Hence $F'_s(0)$ exists.

$$\frac{F(0+h) - F(0)}{h} = \frac{h \sin \frac{1}{h} - 0}{h}$$

$$= \sin \frac{1}{h}$$

$$= \begin{cases} 1 & \text{if } \frac{1}{h} = \frac{(4n+1)\pi}{2} \\ -1 & \text{if } \frac{1}{h} = \frac{(4n+3)\pi}{2} \end{cases}$$

$\therefore \lim_{h \rightarrow 0} \sin \frac{1}{h}$ does not exist.

and F is not differentiable at $x = 0$.

(7 marks)

(c) $\frac{(f+g)(x+h) - (f+g)(x-h)}{2h}$

$$= \frac{f(x+h) - f(x-h) + g(x+h) - g(x-h)}{2h}$$

Since $f'_1(x)$ and $g'_2(x)$ exist,

$$\lim_{h \rightarrow 0} \frac{(f+g)(x+h) - (f+g)(x-h)}{2h} \text{ exists.}$$

5. (c)

and is equal to $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x-h)}{2h}$

i.e. $(f+g)_s(x) = f_s(x) + g_s(x)$

$$\frac{(fg)(x+h) - (fg)(x-h)}{2h} = \frac{f(x+h)g(x+h) - f(x-h)g(x-h)}{2h}$$

$$= g(x+h) \left[\frac{f(x+h) - f(x-h)}{2h} \right] + f(x-h) \left[\frac{g(x+h) - g(x-h)}{2h} \right]$$

Since $f_s(x)$ and $g_s(x)$ exist and f, g are continuous at x ,

$$(fg)_s(x) = g(x)f_s(x) + f(x)g_s(x).$$

(6 marks)

6. (b) (i) For $k = 0, 1, 2, \dots, (n-1)$, let $f(x) = x^k$.

By (a)(ii)

$$\int_{-1}^1 P_n(x) x^k dx = (-1)^k \int_{-1}^1 \left[\frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n \right] \frac{d^k}{dx^k} f(x) dx$$

$$= (-1)^k (k!) \int_{-1}^1 \frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n dx$$

$$= 0 \text{ by (a)(i)}$$

(3 marks)

(ii) $P_m(x)$ is a polynomial of degree m .

$$\text{Let } P_m(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$$

$$\begin{aligned} \therefore I &= \int_{-1}^1 P_n(x) P_m(x) dx = \int_{-1}^1 P_n(x) [a_0 x^m + a_1 x^{m-1} + \dots + a_m] dx \\ &= \sum_{j=0}^m \int_{-1}^1 P_n(x) a_j x^{m-j} dx \end{aligned}$$

If $m < n$, this is equal to zero by (b)(i).

The case for $m > n$ is similar.

(6 marks)

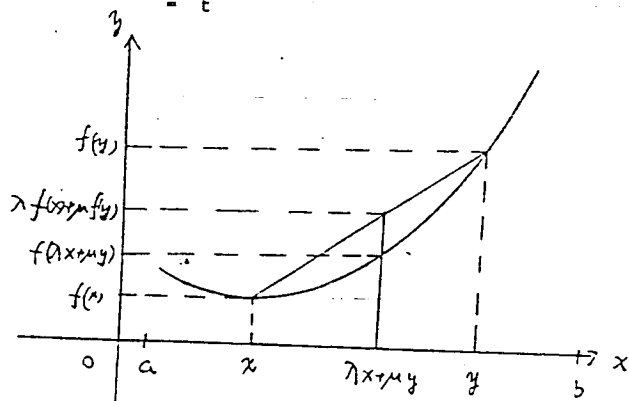
7. (a) For $x < t < y$, let

$$\lambda = \frac{y-t}{y-x}, \quad \mu = \frac{t-x}{y-x}$$

Then $\lambda + \mu = 1$ and $\lambda, \mu \in (0, 1)$.

$$\lambda x + \mu y = \frac{y-t}{y-x}x + \frac{t-x}{y-x}y$$

$$= t$$



4 marks

(b) Let x, t, y be in (a, b) such that $x < t < y$.

$$\begin{aligned} \text{By (a), } f(t) &= f(\lambda x + \mu y) \\ &\leq \lambda f(x) + \mu f(y) \\ &= \frac{y-t}{y-x}f(x) + \frac{t-x}{y-x}f(y) \end{aligned}$$

$$\begin{aligned} (y-x)f(t) &\leq (y-t)f(x) + (t-x)f(y) \quad (\because y-x > 0) \\ (y-t)f(t) - (y-t)f(x) &\leq (t-x)f(y) - (x-t)f(t) \end{aligned}$$

$$\therefore \frac{f(t) - f(x)}{t-x} \leq \frac{f(y) - f(t)}{y-t} \quad (\text{As } t-x, y-t > 0)$$

4 marks

$$\begin{aligned} \text{(c) } h(t) &= \lambda g(t) + \mu g(y) - g(\lambda t + \mu y) \\ h'(t) &= \lambda g'(t) - \lambda g'(\lambda t + \mu y) \end{aligned}$$

Since $g'' \geq 0$ on (a, b) , g' is increasing

$$\begin{aligned} \therefore \lambda t + \mu y > t &\Rightarrow g'(\lambda t + \mu y) \geq g'(t) \\ \therefore h'(t) &\leq 0 \end{aligned}$$

i.e. h is monotonically decreasing.

$$\begin{aligned} \text{Now } h(y) &= \lambda g(y) + \mu g(y) - g(\lambda y + \mu y) \\ &= (\lambda + \mu)g(y) - g((\lambda + \mu)y) \\ &= 0 \quad \text{as } \lambda + \mu = 1. \end{aligned}$$

$$\therefore \lambda g(t) + \mu g(y) - g(\lambda t + \mu y) \leq 0$$

i.e. g is convex.

6 marks

(d) Let $g(x) = x^p$ in (c), where $p > 1, x > 0$.

$$g'(x) = px^{p-1}$$

$$g''(x) = p(p-1)x^{p-2} \geq 0$$

$\therefore g$ is convex.

$$\text{By (a) } g(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 g(x_1) + \lambda_2 g(x_2)$$

$$\text{i.e. } (\lambda_1 x_1 + \lambda_2 x_2)^p \leq \lambda_1 x_1^p + \lambda_2 x_2^p$$

3 marks

(a) Let $P = \left(\frac{p}{\sqrt{2}}, \frac{p}{\sqrt{2}} \right)$
 $Q = \left(\frac{1}{p\sqrt{2}}, -\frac{1}{p\sqrt{2}} \right)$
 $P' = \left(\frac{p'}{\sqrt{2}}, \frac{p'}{\sqrt{2}} \right)$
 $Q' = \left(\frac{1}{p'\sqrt{2}}, -\frac{1}{p'\sqrt{2}} \right)$

Equation of PQ is $y - \frac{p}{\sqrt{2}} = \frac{p^2 + 1}{p^2 - 1} \left(x - \frac{p}{\sqrt{2}} \right)$

Similarly, equation of P'Q' is $y - \frac{p'}{\sqrt{2}} = \frac{p'^2 + 1}{p'^2 - 1} \left(x - \frac{p'}{\sqrt{2}} \right)$

Solving the above

$$u = \frac{pp' + 1}{\sqrt{2}(p + p')}$$

$$v = \frac{pp' - 1}{\sqrt{2}(p + p')}$$

$$\lim_{p' \rightarrow p} u = \frac{p^2 + 1}{2\sqrt{2}p}$$

$$\lim_{p' \rightarrow p} v = \frac{p^2 - 1}{2\sqrt{2}p}$$

7 marks

(b) Eliminating p from $\begin{cases} x = \frac{p^2 + 1}{2\sqrt{2}p} \\ y = \frac{p^2 - 1}{2\sqrt{2}p} \end{cases}$

$$x^2 - y^2 = \frac{1}{8p^2} [(p^2 + 1)^2 - (p^2 - 1)^2]$$

$$= \frac{1}{2}$$

Since $\left(\frac{1}{\sqrt{2}}(p), \frac{1}{\sqrt{2}}(p) \right)$ lies on the first and fourth quadrants, its locus is a branch of the hyperbola.

$$(H) : \frac{x^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{2}} = 1$$

3 marks

8. (c) Let OA = a, OB = b.

Equation of AB is $y - \frac{a}{\sqrt{2}} = \frac{a+b}{a-b} \left(x - \frac{a}{\sqrt{2}} \right)$

Substituting $x = \left(y - \frac{a}{\sqrt{2}} \right) \frac{a-b}{a+b} + \frac{a}{\sqrt{2}}$ in H

$$\left[\left(y - \frac{a}{\sqrt{2}} \right) \frac{a-b}{a+b} + \frac{a}{\sqrt{2}} \right]^2 - y^2 = \frac{1}{2}$$

$$\frac{-8ab}{(a+b)^2} y^2 + \frac{(a-b)8ab}{(a+b)^2} y + \frac{4a^2b^2 - (a+b)^2}{(a+b)^2} = 0$$

The discriminant = $\frac{(a-b)^2 64a^2b^2}{(a+b)^4} + \frac{32ab}{(a+b)^2} \frac{4a^2b^2 - (a+b)^2}{(a+b)^2}$
 $= \frac{32ab}{(a+b)^4} [(a-b)^2 ab + 4a^2b^2 - (a+b)^2]$
 $= \frac{32ab}{(a+b)^4} (ab - 1)(a+b)^2$

Since the discriminant $\begin{cases} < \\ > \end{cases} 0$ according as $ab \begin{cases} < \\ > \end{cases} 1$, the line AB meets (H) at no point, one point, or 2 points according as $ab \begin{cases} < \\ > \end{cases} 1$.

7 marks