

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八一年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION 1981

純數學

試卷一

三小時完卷

上午九時至正午十二時

本試卷必須用英文作答

PURE MATHEMATICS

PAPER I

Three hours

9.00 a.m.—12.00 noon

This paper must be answered in English

This paper consists of eight questions all carrying equal marks.

Answer any SIX questions.

(a) Solve the following system of equations :

$$(I) \begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \end{cases}$$

(b) Find all possible values of p and q such that the following system of equations is solvable :

$$(II) \begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \\ x + y + pz = q \end{cases}$$

(c) Find the solutions, if possible, of the system of equations

$$(III) \begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \\ x + y + pz = 1 \\ x^2 + y^2 + z^2 = 11 \end{cases}$$

2. Let $\{a_n\}$ be a sequence of real numbers. It is known that, if
- $\{a_n\}$ satisfies $a_n < a_{n+1}$ for all n , and
 - there exists a real number K such that $a_n < K$ for all n ,
- then $\{a_n\}$ converges.

- (a) Show that a sequence $\{b_n\}$ of real numbers is convergent if
- $b_n > b_{n+1}$ for all n , and
 - there exists a real number M such that $b_n > M$ for all n .

- (b) Given two positive real numbers a and b such that $a < b$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences satisfying

$$x_1 = a, \quad y_1 = b, \quad \text{and}$$

$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}$$

for all positive integers n .

Show that both $\{x_n\}$ and $\{y_n\}$ converge and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n.$$

3. (a) In how many ways can three different numbers be selected from the thirty numbers $1, 2, \dots, 30$ such that their sum is
- divisible by 2,
 - divisible by 3?

- (b) Using the binomial theorem

$$(x_1 + x_2)^N = \sum_{r=0}^N C_r^N x_1^{N-r} x_2^r$$

for all positive integers N , prove that

$$(y_1 + y_2 + \dots + y_n)^N = \sum_{j_1 + j_2 + \dots + j_n = N} \left[\frac{N!}{j_1! j_2! \dots j_n!} y_1^{j_1} y_2^{j_2} \dots y_n^{j_n} \right]$$

for all positive integers n and N .

4. Given any two complex numbers u and v .

- (a) Prove the inequality $|u + v| \leq |u| + |v|$.

- (b) Prove that the following three statements are equivalent:

$$S_1: |u + v| = |u| + |v| \text{ or } |u - v| = |u| + |v|,$$

$$S_2: \operatorname{Im}(u\bar{v}) = 0,$$

$$S_3: u\bar{v} = \bar{u}v,$$

where $\operatorname{Im}(u\bar{v})$ denotes the imaginary part of $u\bar{v}$.

- (c) If both $|u + v| \neq |u| + |v|$ and $|u - v| \neq |u| + |v|$, prove that, for any complex number z , there exist real numbers a and b such that $z = au + bv$.

5. Let A be a set. Suppose f is a mapping from A into itself such that f is injective but not surjective. Denote the set $\{f(a) : a \in A\}$ by $f[A]$.

- (a) Show that

$$(i) A \neq \emptyset,$$

$$(ii) A \text{ consists of more than two elements.}$$

- (b) If $\varphi : A \rightarrow A$ is a mapping such that

$$\varphi(f(x)) = x$$

for all x in A , show that φ is not bijective.

- (c) Prove that there exists a unique mapping $g : f[A] \rightarrow A$ such that

$$(i) g(f(x)) = x \text{ for all } x \text{ in } A, \text{ and}$$

$$(ii) g \text{ is bijective.}$$

6. For any 2×2 real matrix $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, let $|A| = xw - yz$.

Let

$$S = \left\{ A : |A| \neq 0 \right\} \text{ and}$$

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x \neq 0 \right\}.$$

- (a) Does G form a group under the usual multiplication of matrices? Prove your assertion.

- (b) A relation \sim is defined on S as follows:

$$\text{For } A, B \in S, A \sim B \text{ if there exists } D \in G \text{ such that } AD = B.$$

Show that \sim is an equivalence relation on S .

- (c) Let C be the set of complex numbers. A mapping $\Phi : S \rightarrow C \setminus \{0\}$ is defined by

$$\Phi(A) = \frac{x}{|A|} + i \frac{z}{|A|},$$

$$\text{where } A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Show that

$$(i) \Phi \text{ is surjective,}$$

$$(ii) \Phi(A) = \Phi(B) \text{ if and only if } A \sim B \text{ as defined in (b).}$$

純數學
試卷二

三小時完卷
下午二時至下午五時
本試卷必須用英文作答

PURE MATHEMATICS
PAPER II

Three hours
2.00 p.m.—5.00 p.m.
This paper must be answered in English

This paper consists of eight questions each carrying equal marks.
Answer any SIX questions.

7. (a) Let $f(x) = x - 1 - \log_e x$ for all $x > 0$. Find the minimum value of $f(x)$ and show that $\log_e x < x - 1$ for all $x > 0$. For what value of x will the equality hold?

(b) Let x_1, x_2, \dots, x_n and $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive numbers such that $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

Show that

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} < 1.$$

For what values of x_1, x_2, \dots, x_n will the equality hold?

(c) Prove that, if a_1, a_2, \dots, a_n and p_1, p_2, \dots, p_n are positive numbers, then

$$\left(a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} \right)^{\frac{1}{p_1 + p_2 + \dots + p_n}} < \frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}$$

When will the equality hold?

8. (a) Using DeMoivre's theorem, show that

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{r=0}^n (-1)^r C_{2r+1}^{2n+1} (\cot^2 \theta)^{n-r},$$

where n is a positive integer, $0 < \theta < \frac{\pi}{2}$ and C_{2r+1}^{2n+1} are binomial coefficients.

(b) Using (a) and the relations between the coefficients and roots of the equation

$$\sum_{r=0}^n (-1)^r C_{2r+1}^{2n+1} x^{n-r} = 0, \text{ show that}$$

$$\sum_{k=1}^n \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}, \dots \dots \dots (i)$$

and deduce that

$$\sum_{k=1}^n \operatorname{cosec}^2 \frac{k\pi}{2n+1} = \frac{n(2n+2)}{3} \dots \dots \dots (ii)$$

(c) Let $A_n = \sum_{k=1}^n \frac{1}{k^2}$. Using equations (i) and (ii) of (b), show that

$$\frac{\pi^2}{6} \left(\frac{2n}{2n+1} \right) \left(\frac{2n-1}{2n+1} \right) < A_n < \frac{\pi^2}{6} \left(\frac{2n}{2n+1} \right) \left(\frac{2n+2}{2n+1} \right).$$

Hence evaluate $\lim_{n \rightarrow \infty} A_n$.

(It is known that for $0 < \theta < \frac{\pi}{2}$, $\tan \theta > \theta > \sin \theta$.)

END OF PAPER

1. (a) Find the integrals

(i) $\int \sin(\log_e x) dx$ by parts,

(ii) $\int \frac{dx}{x + \sqrt{x^2 + 1}}$ by using the substitution $t = x + \sqrt{x^2 + 1}$.

(b) Let $F(u) = \int_0^u f(t) dt$, where $f(t)$ is a continuous function.

Show that

$$\int_0^x F(u^2) du = \int_0^{x^2} (x - \sqrt{u}) f(u) du$$

for all $x > 0$.

(c) Find the area bounded by the curve $2y^2 + x - 3 = 0$ and the straight line $2y + x + 1 = 0$.

2. Given two planes $\pi_1 : x + y - z - 1 = 0$,
 $\pi_2 : x - y = 0$.

(a) Find the foot F of the perpendicular from the point $P(2, 2, -1)$ to the plane π_1 .

(b) Find two points Q and R in the plane π_2 such that PQR is an equilateral triangle with the line segment PF as a median.

3. Suppose that $A(x_1, y_1)$ and $B(x_2, y_2)$ are distinct points lying on the parabola $8y^2 = x$.

(a) Show that there is a point $C(x_3, y_3)$ lying on the parabolic arc AB such that the tangent at C is parallel to the chord joining A and B .

Express x_3 and y_3 in terms of y_1 and y_2 .

(b) Let $h = y_2 - y_1$. Find the area of $\triangle ABC$ in terms of h .

(c) Show that the circle passing through the points A , B and C will intersect the parabola again at a point $D(x_4, y_4)$.

Express x_4 and y_4 in terms of y_1 and y_2 .

4. (a) Resolve $\frac{1}{(1+x)(1+2x)\dots(1+nx)}$ into partial fractions.

(b) Use the result in (a) to prove the identity

$$\sum_{r=0}^n (-1)^{n-r} C_r^n r^n = n!$$

where C_r^n are binomial coefficients.

(c) Prove that the n th derivative with respect to t of $(e^t - 1)^n$ takes the value $n!$ when t is zero.

5. (a) Suppose that a function $f(x)$ is increasing on the interval $\{x : x \geq 1\}$.

Show that

$$\sum_{i=1}^k f(i) < \int_1^{k+1} f(x) dx < \sum_{i=2}^{k+1} f(i),$$

where k is a positive integer.

Hence show that

$$\log_e[(n-1)!] < \int_1^n \log_e x dx < \log_e(n!),$$

and that

$$(n-1)! < n^n e^{-n+1} < n!$$

where n is a positive integer.

(b) Prove that $\lim_{n \rightarrow \infty} \left\{ (a+n)^{\frac{1}{n}} - 1 \right\} = 0$, where a and b are positive numbers.

(c) Using the results in (a) and (b), or otherwise, find $\lim_{n \rightarrow \infty} \frac{(n!)^{\frac{1}{n}}}{n}$.

6. Given two non-constant functions $\varphi(x)$ and $\lambda(t)$ satisfying the relation

$$\varphi(x+t) + \varphi(x-t) - 2\varphi(x) = \lambda(t)$$

for all real numbers x and t .

(a) Prove that the function $\varphi(x)$ cannot have both an absolute maximum and an absolute minimum.

(b) If $\varphi(x)$ is differentiable for all real x , prove that

$$\varphi'(x+y) - \varphi'(x) = \varphi'(y) - \varphi'(0)$$

for all x and y .

(c) Suppose, further, that $\varphi''(0)$ exists. Prove that $\varphi''(x)$ exists for all x and that $\varphi''(x) = \varphi''(0)$.

Hence show that $\varphi(x)$ is a polynomial of degree less than or equal to two.

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試卷一

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PURE MATHEMATICS
PAPER I

Three hours
9.00 a.m.—12.00 noon
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This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

7. Suppose that the function $f(x)$ is continuous for all $x > 0$ and $\lim_{x \rightarrow \infty} f(x) = l$ exists.

(a) Show that

$$\int_a^b \frac{f(rx) - f(sx)}{x} dx = \int_{ra}^{rb} \frac{f(x)}{x} dx - \int_{sa}^{sb} \frac{f(x)}{x} dx,$$

where $0 < a < b$ and $0 < r < s$.

(b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers such that

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = +\infty.$$

Show that

$$\lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x} dx = (f(0) - l) \log_e \left(\frac{s}{r}\right).$$

[Hint: you may assume without proof that the following theorem holds:

If $g(x)$ and $h(x)$ are continuous on the closed interval $[c, d]$ and $h(x) > 0$ for all x in $[c, d]$, then

$$\int_c^d g(x)h(x)dx = g(x_0) \int_c^d h(x)dx$$

for some x_0 in $[c, d]$.]

(c) Furthermore, if $f(x) > c$ for all $x > 0$, where c is a positive constant, is it then true that

$$\lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x} dx = \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{f(rx)}{x} dx - \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{f(sx)}{x} dx?$$

Prove your assertion.

8. Suppose that $f(x) = \int_1^x \frac{1}{\sqrt{1+t^2}} dt$ for $x > 1$.

(a) (i) Show that $f(x) < f(y)$ whenever $1 < x < y$.

(ii) Show that $f(x) < \frac{2}{3}$ for all $x > 1$.

(iii) Find an x_0 such that $f(x_0) > \frac{1}{3}$.

(b) Let $g(u)$, where $0 < u < f(x_0)$, be a function such that $f(g(u)) = u$.

(i) Show that $g'(u) = [1 + g^2(u)]^{\frac{1}{2}}$ and $g''(u) = \frac{5}{2} (g(u))^4$.

(ii) Let $h(u) = e^u - g(u)$, where $0 < u < f(x_0)$. Prove that $h''(u) < h'(u)$.

Hence prove that $h(u)$ does not have a minimum.

END OF PAPER

1. (a) Let $0 < \lambda < 1$. Show that

$$\lambda t + (1 - \lambda) > t^\lambda \quad \text{for all } t > 0.$$

Deduce that

$$\lambda \alpha + (1 - \lambda)\beta > \alpha^\lambda \beta^{1-\lambda} \quad \text{for all } \alpha, \beta > 0.$$

(b) Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two sets of non-negative real numbers such that

$$\sum_{i=1}^n a_i^p = \sum_{i=1}^n b_i^q = 1.$$

Using the result in (a), show that $\sum_{i=1}^n a_i b_i < 1$.

Hence show that, for any two sets of non-negative real numbers $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$,

$$\sum_{i=1}^n x_i y_i < \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q \right)^{\frac{1}{q}}.$$