Section A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(E) answer book.

1. It is given that

$$\left(1+\frac{x}{a}\right)^r = 1+\frac{x}{24}-\frac{x^2}{576}+\text{terms involving higher powers of } x,$$

where r is a rational number and a is a non-zero integer.

(a) Find the values of r and a.

(b) Expand $(8+x)^{\frac{1}{3}}$ in ascending powers of x up to the term involving x^2. State the range of x for which the expansion is valid.

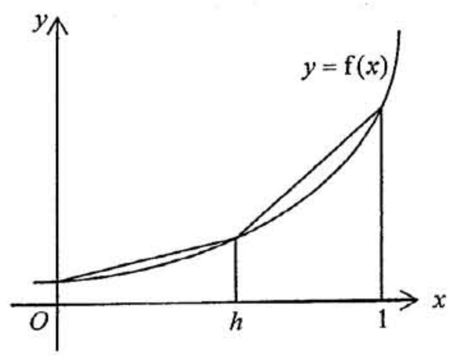
(6 marks)

2. Let $f(x)=e^{2x}$.

(a) Use trapezoidal rule with 2 intervals of equal width to find the approximate value of $\int_0^1 f(x)\,dx$.

(b) Evaluate the exact value of $\int_0^1 f(x)\,dx$.

(c) A student uses trapezoidal rule with 2 trapeziums of unequal widths to approximate $\int_0^1 f(x)\,dx$. The first trapezium has width h $(0<h<1)$ and the second trapezium has width $1-h$ as shown below. Let A be the total area of the two trapeziums.



(i) Show that $A=\dfrac{e^{2h}+(1-e^2)h+e^2}{2}$.

(ii) Find the minimum value of A.

(8 marks)

3. An archaeologist models the presence of carbon-14 remaining in animal skulls fossil by $\frac{dA}{dt} = -kA$ where A (in grams) is the amount of carbon-14 present in the skull at time t (in years) and k is a constant. Let A_0 (in grams) be the original amount of carbon-14 in the skull. It is known that half of the carbon-14 will disappear after 5 730 years.

(a) By expressing $\frac{dt}{dA}$ in terms of A, or otherwise, find the value of k correct to 3 significant figures.

(b) In an animal skull fossil, the archaeologist found that 30% of the original amount of carbon-14 is still present. Find the approximate age of the skull correct to the nearest ten years.

(6 marks)

4. Let A and B be two exhaustive events of a certain sample space. Denote $P(B) = b$ and $P(A \cap B) = c$, where $0 < b < 1$ and $0 < c < 1$.

(a) Express $P(A)$ in terms of b and c.

(b) Suppose that $P(A \mid B) = \frac{1}{2}$ and $P(B \mid A) = \frac{2}{3}$.

(i) Find the values of b and c.

(ii) Are the events A and B independent? Explain your answer.

(7 marks)

5. The following stem-and-leaf diagram shows the distribution of the test scores of 21 students taking a statistics course.

Stem (Tens)	Leaf (Units)
2	a
3	
4	9
5	0 0 1 3 7 7
6	0 2 3 5 5 5 9
7	0 3 4 9
8	2 b

Let $\bar{x}$ be the mean of these 21 scores.
It is known that if the smallest value of these 21 scores is removed, the range is decreased by 27 and the mean is increased by 2.

(a) Find the values of a, b and $\bar{x}$.

(b) The teacher wants to select 6 students to participate in a competition by first excluding the student with the lowest score. If the students are randomly selected, find the probability that there will be

(i) no students with score higher than 70 being selected;

(ii) at least 2 students with scores higher than 70 being selected.

(6 marks)

6. The coach of a girls school basketball team recruits new members from the Form One students, of whom 11.7% are taller than 152 cm . Assume that their heights are normally distributed with a mean of μ cm and a standard deviation of 5 cm .

(a) Find the value of μ .

(b) It is known that 20% of the Form One students taller than 152 cm do not apply to join the basketball team, while 10% of students shorter than 152 cm apply to join.
If a Form One student is selected at random, find the probability that

(i) the student applies to join the basketball team;

(ii) the student is shorter than 152 cm given that she does not apply to join the basketball team.

(7 marks)

Section B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Define $\mathrm{f}(x)=\dfrac{x+4}{x+a}$ where a is a constant and $x \neq -a$. Let C_1 be the curve $y=\mathrm{f}(x)$ with the vertical asymptote $x=2$.

   (a) (i) Find the value of a .

   (ii) Find the equation(s) of the horizontal asymptote(s) of C_1 .

   (iii) Find $\mathrm{f}'(x)$ and hence find the range of x for which $\mathrm{f}(x)$ is decreasing.

   (iv) Sketch the curve of C_1 . Indicate its asymptote(s) and intercept(s).

   (8 marks)

   (b) Let C_2 be the curve $y=\mathrm{g}(x)$, where $\mathrm{g}(x)=\mathrm{f}(x-2)+2$.

   (i) Using the result in (a)(iv), or otherwise, sketch the curve C_2 . Indicate its asymptote(s) and intercept(s).

   (ii) Let C_3 be the curve $y=[\mathrm{g}(x)]^2$. Find the area of the region bounded by the curve C_3 and the axes.

   (7 marks)

8. A company launches a campaign to increase the sales of a product. The monthly increase in sales (in thousand dollars) t months after the launch can be modelled by the function

$$f(t) = -250e^{2at} + 300e^{at} - 50$$

where a is a non-zero constant.
It is known that the monthly increase in sales attains the maximum 5 months after the launch.

(a) Find the value of a.

(3 marks)

(b) After at least T_1 months, the campaign will not increase the sales.

(i) Find the value of T_1.

(ii) Estimate the total amount of sales increased T_1 months after the launch.

(5 marks)

(c) The start up cost of the campaign is 100 thousand dollars and the running cost at time t is $\dfrac{100}{t+9}$ thousand dollars. The campaign will be terminated after T_2 months when the total expenditure reaches 200 thousand dollars.

(i) Express the total expenditure E (in thousand dollars) in terms of t.

(ii) Find the value of T_2.

(iii) During the period of the campaign, the manager of the company suggests replacing the campaign by a less costly plan. The monthly increase in sales (in thousand dollars) due to the plan can be modelled by the function

$$g(t) = -(t-\alpha)(t-2\alpha) \quad , \quad \alpha \le t \le 2\alpha$$

where α $(0 < \alpha \le T_2)$ is the time, in months after the launching of the original campaign, of starting the plan.
In order to achieve the maximum total amount of sales increased by the plan, when should it be started? Explain your answer.

(7 marks)

9. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t) = \frac{a}{b+e^{-t}} + c \quad , \quad -\infty < t < \infty$$

where a, b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a, b and c,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population.*

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a, b and c.

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the *primordial population* and *ultimate population.* Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a, b, c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha < \beta$.

Hence express α and β in terms of a, b and c.

Sketch $p'(t)$ against $p(t)$ for $\alpha < p(t) < \beta$ and hence verify your answer in (c).

(5 marks)

10. The marketing manager of a hotel studies the number of wedding banquets held per week in the hotel for the past 52 weeks. He proposes two Poisson models to explain the observed data in the following table. He uses the sample mean of the observed data to estimate λ_1 .

Number of wedding banquets per week	Observed frequency f_0	* Expected frequency f_e	
		Po(λ_1)	Po(λ_2)
0	7	a	k
1	18	b	12.68
2	12	c	13.94
3	11	8.07	10.23
4	4	d	5.62

* Correct to 2 decimal places.

(a) (i) Find the sample mean of the observed data.

(ii) Find λ_2 correct to 2 decimal places.

(iii) Find the values of a , b , c , d and k correct to 2 decimal places.

(iv) The absolute value of the difference between the observed frequency and the corresponding expected frequency is defined as error. The distribution with a smaller total sum of errors is regarded as fitting the observed data better. Which distribution fits the observed data better? Explain your answer.

(8 marks)

(b) The distribution that fits the observed data better in (a)(iv) is adopted. Assume that the expense per table in a wedding banquet follows a normal distribution with a mean of \$ 7 388 and a standard deviation of \$ 1 200 .

(i) Find the probability that the expense per table is between \$ 6 188 and \$ 8 888 in a certain wedding banquet.

(ii) Given that there are at most 3 wedding banquets in a certain week, find the probability that there is exactly 1 banquet of which the expense per table is between \$ 6 188 and \$ 8 888 .

(7 marks)

11. In a promotion period of an electronic shopping card with spending limit of $3 000, cardholders who spend over $400 in the maximum amount transaction are classified as VIPs and are eligible for entering an online "click-and-get-point" game once. The rules of the game are detailed in the following table.

Spending ($\$x$)	VIP Category	Number of clicks allowed
$400 < x \leq 800$	Silver	1
$800 < x \leq 1000$	Gold	2
$1000 < x \leq 3000$	Platinum	3

The probabilities to get 1, 2, 3 and 4 points on a single click are 0.4, 0.3, 0.2 and 0.1 respectively. The total number of points got in a game can be exchanged for a cash rebate according to the following table.

Total number of points	Cash rebate
1 to 3	$20
4 to 9	$50
10 to 12	$200

It is known that among the VIPs, 25% belong to Silver, 60% belong to Gold and 15% belong to Platinum.

(a) In a certain completed game, find the probability

(i) of getting exactly 3 points if the player is a Gold VIP;

(ii) of getting exactly 3 points;

(iii) that the player is a Gold VIP given that the player gets exactly 3 points.

(5 marks)

(b) Find the probability that the player gets a cash rebate of exactly $ 20 in a certain completed game.

(2 marks)

(c) In a certain completed game, find the probability that the player gets

(i) exactly 10 points;

(ii) a cash rebate of exactly $ 200.

(3 marks)

(d) Research data reveal that 70% of each category of the VIPs will complete the game. A manager of the card company proposes offering a 4% direct cash rebate of the transaction to all VIPs instead of the online game. However, a senior manager, Winnie, thinks that the cost of that proposal will certainly be higher than the expected cash rebate of the online game.

(i) Do you agree with Winnie? Explain your answer.

(ii) Another senior manager, John, thinks that the cost of offering a 2% direct cash rebate to all VIPs will certainly be lower than the expected cash rebate of the online game. Do you agree with John? Explain your answer.

(5 marks)

12. A manufacturer produces a specific kind of tablets. He uses one machine to produce ingredient A and ingredient B, and then one mixer to mix the ingredients to produce the tablets and pack them in bags. The bags of tablets are then delivered to a hospital.

Past records indicate that 0.6% of ingredients A and B respectively are contaminated during the ingredient production process, while 0.1% of the tablets are contaminated during the mixing and packing process. A tablet is regarded as a *contaminated tablet* if

- the ingredient A in the tablet is contaminated, or
- the ingredient B in the tablet is contaminated, or
- the tablet is contaminated during the mixing and packing process.

The pharmacist of the hospital draws a random sample of 20 tablets from each bag to test for contamination. A bag is considered *unsafe* if it contains more than 1 tablet tested positive as a *contaminated tablet*.

(a) Find the probability that a randomly selected tablet from a certain bag is a *contaminated tablet*.
(3 marks)

(b) Find the probability that a bag of tablets is regarded *unsafe*.
(2 marks)

(c) In a certain week, 100 bags of such tablets are delivered to the hospital. The hospital will suspend the supply of the tablets from the manufacturer if more than 4 bags are found *unsafe* within a week.

(i) Find the probability that the 10th bag will be the first one which is regarded *unsafe*.

(ii) Find the probability that the supply from the manufacturer will be suspended in a certain week.
(5 marks)

(d) The manufacturer wants to increase the production and requires the probability of a tablet being contaminated to be less than 1%. To achieve this, he plans to add n new machines for producing the ingredients A and B which has contamination probability of 0.4% respectively. Suppose equal amount of ingredients A and B are produced by the original machine and each of the n new machines.

(i) Express the probability that the ingredient A is contaminated in terms of n.

(ii) What is the least value of n?
(5 marks)

END OF PAPER