評卷參考 *
Marking Scheme

香港考試及評核局
Hong Kong Examinations and Assessment Authority

2007年香港高級程度會考
Hong Kong Advanced Level Examination 2007

數學及統計學 高級補充程度
Mathematics and Statistics AS-Level

This document was prepared for markers’ reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

*此部分只設英文版本
AS Mathematics and Statistics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits all the marks allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

- ‘M’ marks awarded for correct methods being used;
- ‘A’ marks awarded for the accuracy of the answers;
- Marks without ‘M’ or ‘A’ awarded for correctly completing a proof or arriving at an answer given in a question.

In a question consisting of several parts each depending on the previous parts, ‘M’ marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, ‘A’ marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates’ work. In general, marks for a certain step should be awarded if candidates’ solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates’ work, the benefit of doubt should be given in the candidates’ favour.

6. Marks may be deducted for poor presentation (pp). The symbol \( pp \) should be used to denote 1 mark deducted for \( pp \). At most deducted 1 mark from Section A and 1 mark from Section B for \( pp \). In any case, do not deduct any marks for \( pp \) in those steps where candidates could not score any marks.

7. Marks may be deducted for numerical answers with inappropriate degree of accuracy \( (a) \). The symbol \( a \) should be used to denote 1 mark deducted for \( a \). At most deducted 1 mark from Section A and 1 mark from Section B for \( a \). In any case, do not deduct any marks for \( a \) in those steps where candidates could not score any marks.

8. In the marking scheme, ‘r.t.’ stands for ‘accepting answers which can be rounded off to’ and ‘f.t.’ stands for ‘follow through’. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

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1. (a) (i) \[ \left(1 + \frac{x}{a}\right)^r = 1 + \frac{r}{a} x + \frac{r(r-1)}{2!} \left(\frac{x}{a}\right)^2 + \ldots \]
= \[1 + \frac{rx}{a} + \frac{r(r-1)}{2a^2} x^2 + \ldots \]
So, we have \( \frac{r}{a} = \frac{2r}{3} \) and \( \frac{r(r-1)}{2a^2} = -\frac{1}{18} \).
Solving, we have \( a = \frac{3}{2} \) and \( r = \frac{1}{2} \).

(ii) The binomial expansion is valid for \( \left| \frac{2x}{3} \right| < 1 \).
Thus, the range of values of \( x \) is \( -\frac{3}{2} < x < \frac{3}{2} \).

(b) (i) \( \left(1 - \frac{x}{a}\right)^r = 1 - \frac{1}{3} x - \frac{1}{18} x^2 + \ldots \)

(ii) The binomial expansion is valid for \( \left| \frac{-2x}{3} \right| < 1 \).
Thus, the range of values of \( x \) is \( -\frac{3}{2} < x < \frac{3}{2} \).

2. (a) Let \( u = 2t^2 + 50 \).
Then, we have \( \frac{du}{dt} = 4t \).

\[ N = \int \frac{800t}{(2t^2 + 50)^2} \, dt \]
\[ = \int \frac{200}{u^2} \, du \]
So, we have \( N = -\frac{200}{u} + C \), where \( C \) is a constant.

Therefore, we have \( N = -\frac{200}{2t^2 + 50} + C \).
Using the condition that \( N = 4 \) when \( t = 0 \), we have \( 4 = -4 + C \).
Hence, we have \( C = 8 \).
Thus, we have \( N = 8 - \frac{200}{2t^2 + 50} \).
### Alternative Solution

Let \( u = 2v^2 + 50 \).

Then, we have \( du = 4vdv \).

\[
\left[ N \right]_0^t = \int_0^t \frac{800v}{u^2} dv = \int_{50}^{2t^2+50} \frac{200}{u^2} du = \left[ \frac{-200}{u} \right]_{50}^{2t^2+50}
\]

\[ \therefore N - 4 = \frac{-200}{2t^2 + 50} - \frac{-200}{50} \]

i.e. \( N = 8 - \frac{200}{2t^2 + 50} \).

1. **Let \( u = 2v^2 + 50 \).**

2. **Integration by substitution.**

3. **For simplifying the integral.**

4. **Using the initial condition \( N(0) = 4 \).**

(b) When \( N = 6 \), we have \( 8 - \frac{200}{2t^2 + 50} = 6 \).

So, we have \( t = 5 \).

The number of bacteria will be 6 million 5 days after the start of the research.

3. **(a) \( y = \frac{1 - e^{4x}}{1 + e^{8x}} \)**

\[
\frac{dy}{dx} = \frac{(1 + e^{8x})(-4e^{4x}) - (1 - e^{4x})(8e^{8x})}{(1 + e^{8x})^2}
\]

When \( x = 0 \), we have \( \frac{dy}{dx} = -2 \).

1. **For the quotient rule.**

2. **Product rule.**

### Alternative Solution

\[
y = \frac{1 - e^{4x}}{1 + e^{8x}}
\]

\[
\ln y = \ln(1 - e^{4x}) - \ln(1 + e^{8x})
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{-4e^{4x}}{1 - e^{4x}} - \frac{8e^{8x}}{1 + e^{8x}}
\]

\[
\frac{dy}{dx} = \frac{1 - e^{4x}}{1 + e^{8x}} \left( \frac{-4e^{4x}}{1 - e^{4x}} - \frac{8e^{8x}}{1 + e^{8x}} \right)
\]

\[
= \frac{-4e^{4x}}{1 + e^{8x}} \left( \frac{8(1 - e^{4x})e^{8x}}{(1 + e^{8x})^2} \right)
\]

When \( x = 0 \), \( \frac{dy}{dx} = \frac{-4}{1 + 1} = -2 \).
(b) (i) Since \((z^2 + 1)e^{3z} = e^{\alpha x + \beta x}\), we have \(\ln(z^2 + 1) + 3z = \alpha + \beta x\).

(ii) Since the graph of the linear function passes through the origin and the slope of the graph is 2, we have \(\alpha = 0\) and \(\beta = 2\).

(iii) \(\ln(z^2 + 1) + 3z = 2x\)
\[
\frac{2x}{z^2 + 1} = 2 \frac{dx}{dz}
\]
Therefore, we have \(\frac{dx}{dz}_{z=0} = \frac{3}{2}\).

Note that \(x = 0\) when \(z = 0\).

Also note that \(\frac{dy}{dx}_{x=0} = -2\).

\[
\frac{dy}{dz}_{z=0} = \left( \frac{dy}{dx}_{x=0} \right) \left( \frac{dx}{dz}_{z=0} \right) = (-2) \left( \frac{3}{2} \right) = -3
\]

\[
y = \frac{1 - e^{6x+2\ln(z^2+1)}}{1 + e^{12x+4\ln(z^2+1)}}
\]
\[
y = \frac{1 - (z^2 + 1)^2 e^{6z}}{1 + (z^2 + 1)^4 e^{12z}}
\]
\[
\frac{dy}{dz} = \frac{-\left[1 - (z^2 + 1)^2 e^{6z}\right] \left[12(z^2 + 1)^4 e^{12z} + 4(z^2 + 1)^3(2z)e^{12z}\right]}{\left[1 + (z^2 + 1)^4 e^{12z}\right]^2}
\]
\[
\frac{dy}{dz}_{z=0} = -3
\]

1A for both correct

1A

1M for chain rule

1A

1M for quotient rule or product rule

1A

---------(7)
4. (a) (i) Note that \( 5.1 < 5.3 \) and \( k \geq 0 \).

Hence, we have \( 5.3 = k - 1.2 \).

Thus, we have \( k = 6.5 \).

\[
5.3 = k - 1.2 \quad \text{or} \quad 5.3 = 5.1 - k
\]

So, we have \( k = 6.5 \) or \( k = -0.2 \) (rejected as \( k \geq 0 \)).

Thus, we have \( k = 6.5 \).

(ii) \hspace{1cm} \begin{array}{|c|c|c|c|c|c|}
\hline
\text{Stem (units)} & \text{Leaf (tenths)} \\
\hline
1 & 2 & 8 & 9 \\
2 & 1 & 1 & 2 & 3 & 4 & 4 & 9 \\
3 & 6 & 7 & 9 \\
4 & 7 \\
5 & 1 \\
6 & 5 \\
\hline
\end{array}
\]

\(1M + 1A\)

(iii) The mean

\[
= 3.05 \text{ hours}
\]

The median

\[
= 2.4 \text{ hours}
\]

(b) The revised mean is greater than the mean obtained in (a)(iii).

The revised median is the same as the median obtained in (a)(iii).

The change in the mean is positive.

There is no change in the median.

\(1A\)

\(1A\)

\(1A\)

\(1A\)

\(---------(7)\)
5. (a) \(P(A' \cap B)\)
\[= P(B | A') P(A')\]
\[= 0.3(1 - a)\]
\(P(A' \cap B)\)
\[= P(A' | B) P(B)\]
\[= 0.6b\]
Hence, we have \(0.6b = 0.3(1 - a)\).
Thus, we have \(a + 2b = 1\).

(b) \(P(A \cap B')\)
\[= P(B' | A) P(A)\]
\[= 0.7a\]
\(P(A \cup B')\)
\[= 1 - P(A' \cap B)\]
\[= 1 - 0.6b\]
Note that \(P(A \cup B') = P(A) + P(B') - P(A \cap B')\).
Hence, we have \(1 - 0.6b = a + (1 - b) - 0.7a\).
So, we have \(3a = 4b\).
Solving \(a + 2b = 1\) and \(3a = 4b\), we have \(a = 0.4\) and \(b = 0.3\).

(c) Since \(P(A \cap B) = P(A) - P(A \cap B')\), \(P(A) = 0.4\) and \(P(A \cap B') = 0.28\), we have \(P(A \cap B) = 0.12 = (0.4)(0.3) = P(A)P(B)\).
Thus, \(A\) and \(B\) are independent events.

Since \(P(A) = a\), we have \(P(A \cap B) = P(A) - P(A \cap B') = a - 0.7a = 0.3a\).
With the help of \(P(B) = 0.3\), we have \(P(A \cap B) = P(A)P(B)\).
Thus, \(A\) and \(B\) are independent events.

Since \(P(A' | B) = 0.6\), we have \(P(A | B) = 1 - P(A' | B) = 1 - 0.6 = 0.4\).
With the help of \(P(A) = 0.4\), we have \(P(A | B) = P(A)\).
Thus, \(A\) and \(B\) are independent events.
<table>
<thead>
<tr>
<th>Solution</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. (a)</td>
<td></td>
</tr>
</tbody>
</table>
| The required probability  
\[ = \left( \frac{1}{10} \right)^3 \left( \frac{3}{8} \right)^4 \]  
\[ = \frac{1}{60} \]  
\[ \approx 0.016666667 \]  
\[ \approx 0.0167 \]  
| 1A for numerator or denominator  
+ 1A for all  
1A  
a−1 for r.t. 0.017 |
| The required probability  
\[ = \left( \frac{C_4^1 C_6^3}{C_6^0 C_2^3} \right) \left( \frac{1}{C_2^1} \right) \]  
\[ = \frac{1}{60} \]  
\[ \approx 0.016666667 \]  
\[ \approx 0.0167 \]  
| 1A for 1st bracket + 1A for all  
1A  
a−1 for r.t. 0.017 |
| The required probability  
\[ = \left( \frac{C_4^1}{C_2^0} \right) \left( \frac{C_6^3}{C_2^9} \right) \]  
\[ = \frac{1}{60} \]  
\[ \approx 0.016666667 \]  
\[ \approx 0.0167 \]  
| 1A for 2nd bracket + 1A for all  
1A  
a−1 for r.t. 0.017 |
| The required probability  
\[ = \frac{P_1^3 P_2^4}{P_0^{10}} \]  
\[ = \frac{1}{60} \]  
\[ \approx 0.016666667 \]  
\[ \approx 0.0167 \]  
| 1A for numerator or denominator  
+ 1A for all  
1A  
a−1 for r.t. 0.017 |
| (b)  |       |
| The required probability  
\[ = \frac{1}{60} + \frac{\binom{5}{2} \binom{5}{4}}{10 \cdot 9 \cdot 8} \]  
\[ = \frac{1}{45} \]  
\[ \approx 0.155555556 \]  
\[ \approx 0.1556 \]  
| 1M for \((p + q + r + s)\) + 1M for using (a)  
1A  
a−1 for r.t. 0.156 |
| The required probability  
\[ = \frac{1}{60} + \frac{C_4^1 C_6^3}{C_6^0 C_2^9} \]  
\[ = \frac{1}{45} \]  
\[ \approx 0.155555556 \]  
\[ \approx 0.1556 \]  
| 1M for \((p + q + r + s)\) + 1M for using (a)  
1A  
a−1 for r.t. 0.156 |
<table>
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<th>Solution</th>
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</thead>
<tbody>
<tr>
<td>The required probability</td>
<td>1M for ((p + q)) + 1M for using (a)</td>
</tr>
<tr>
<td>[\frac{1}{60} + \frac{r_1^5 \cdot r_2^5}{p_1^{10}}]</td>
<td>1A</td>
</tr>
<tr>
<td>= \frac{7}{45}</td>
<td>a-1 for r.t. 0.156</td>
</tr>
<tr>
<td>(\approx 0.1555555556)</td>
<td></td>
</tr>
<tr>
<td>(\approx 0.1556)</td>
<td></td>
</tr>
<tr>
<td>The required probability</td>
<td>1M for ((p + q + r + s))</td>
</tr>
<tr>
<td>[\frac{4}{10} \cdot \frac{3}{9} \cdot \frac{5}{8} + \frac{4}{10} \cdot \frac{2}{9} \cdot \frac{5}{8}]</td>
<td>2A</td>
</tr>
<tr>
<td>= \frac{7}{45}</td>
<td></td>
</tr>
<tr>
<td>(\approx 0.1555555556)</td>
<td></td>
</tr>
<tr>
<td>(\approx 0.1556)</td>
<td>a-1 for r.t. 0.156</td>
</tr>
<tr>
<td>The required probability</td>
<td>1M for ((p + q + r + s))</td>
</tr>
<tr>
<td>[\frac{C_r^4 \cdot C_1^6}{C_1^6 \cdot C_1^8} + \frac{C_1^4 \cdot C_1^6}{C_2^6 \cdot C_1^8}]</td>
<td>2A</td>
</tr>
<tr>
<td>= \frac{7}{45}</td>
<td></td>
</tr>
<tr>
<td>(\approx 0.1555555556)</td>
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<td>a-1 for r.t. 0.156</td>
</tr>
<tr>
<td>The required probability</td>
<td>1M for ((p + q + r + s))</td>
</tr>
<tr>
<td>[\frac{p_2^4 \cdot p_3^6}{p_1^{10}} + \frac{p_1^2 \cdot p_4^4 \cdot p_5^5}{p_1^{10}}]</td>
<td>2A</td>
</tr>
<tr>
<td>= \frac{7}{45}</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>(\approx 0.1556)</td>
<td>a-1 for r.t. 0.156</td>
</tr>
</tbody>
</table>

\[\text{(6)}\]
Solution

7. (a) \[ \lim_{x \to \infty} \frac{8x - 40}{x + 4} = \lim_{x \to \infty} \frac{8}{1 + \frac{4}{x}} = 8 \]

\[ \Rightarrow \text{the equation of the horizontal asymptote to } C_1 \text{ is } y = 8. \]

\[ \Rightarrow \text{the equation of the vertical asymptote to } C_1 \text{ is } x = -4. \]

The x-intercept of \( C_1 \) is 5.
The y-intercept of \( C_1 \) is -10.

Marks

1A for all the asymptotes of \( C_1 \)
1A for all the intercepts of \( C_1 \)
1A for the shape of \( C_1 \)
(b) (i) \( g(x) = \frac{(x+4)^2(x+5)}{8} \)
\( g'(x) = \frac{3(x+4)(x-2)}{8} \)
\( g'(x) = 0 \) when \( x \neq -4 \) or \( x = 2 \)
\( g''(x) = \frac{3(x+1)}{4} \)
\( g''(x) = 0 \) when \( x = -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x &lt; -4 )</th>
<th>( -4 )</th>
<th>(-4 &lt; x &lt; -1 )</th>
<th>(-1 &lt; x &lt; 2 )</th>
<th>2</th>
<th>( x &gt; 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g'(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( g''(x) )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>( \searrow )</td>
<td>0</td>
<td>( \Uparrow )</td>
<td>( -\frac{27}{4} )</td>
<td>( \Uparrow )</td>
<td>( -\frac{27}{2} )</td>
</tr>
</tbody>
</table>

Since \( g'(2) = 0 \) and \( g''(2) > 0 \),
the coordinates of the relative minimum point are \( (2, -\frac{27}{2}) \).

Since \( g'(-4) = 0 \) and \( g''(-4) < 0 \),
the coordinates of the relative maximum point are \( (-4, 0) \).

Since \( g''(x) \left\{ \begin{array}{ll}
< 0 & \text{if} \quad x < -1 \\
0 & \text{if} \quad x = -1 \\
> 0 & \text{if} \quad x > -1
\end{array} \right. \)
the coordinates of the point of inflexion are \( (-1, -\frac{27}{4}) \).

(ii) \( C_1 : y = f(x) \), where \( f(x) = \frac{8x - 40}{x + 4} \).
\( C_2 : y = g(x) \), where \( g(x) = \frac{(x+4)^2(x-5)}{8} \).

Note that \( f(x) = g(x) \)
\( \iff \frac{8x - 40}{x + 4} = \frac{(x+4)^2(x-5)}{8} \)
\( \iff 64(x-5) = (x+4)^3(x-5) \)
\( \iff (x-5)((x+4)^3 - 64) = 0 \)
\( \iff x = 0 \) or \( x = 5 \)
So, the coordinates of the points of intersection are \( (0, -10) \) and \( (5, 0) \).

Also, the \( y \)-intercepts of \( C_1 \) and \( C_2 \) are \(-10\).

When \( f(x) = 0 \), we have \( x = 5 \).
When \( g(x) = 0 \), we have \( x \neq -4 \) or \( x = 5 \).
So, the \( x \)-intercept of \( C_1 \) is \( 5 \).
Also, the \( x \)-intercepts of \( C_2 \) are \(-4 \) and \( 5 \).
Solution

(c) The required area

\[ = \int_0^4 (f(x) - g(x)) \, dx \]

\[ = \int_0^4 \left( \frac{8x - 40}{x + 4} \right) \, dx \]

\[ = \int_0^4 \left( 8 - \frac{72}{x + 4} \right) \, dx \]

\[ = \left[ 8x - 72 \ln(x+4) - \frac{x^3}{8} + \frac{3x^2}{8} - \frac{3x}{2} - 10x \right]_0^4 \]

\[ = \left[ \frac{2955}{32} - 72 \ln \left( \frac{9}{4} \right) \right] - \left[ \frac{2955}{32} - 144 \ln \left( \frac{3}{2} \right) \right] \]

\[ \approx 33.95677443 \]

\[ \approx 33.9568 \]

---

1A for the shape of \( C_2 \)
1A for all the extreme points and the point of inflexion
1A for all the points of intersection

1M accept \( \int_0^4 (g(x) - f(x)) \, dx \)

1M for division

1A for correct integration

1A

\( \sigma - 1 \) for r.t. 33.957

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<table>
<thead>
<tr>
<th>Solution</th>
<th>Marks</th>
</tr>
</thead>
</table>
| 8. (a) (i) The total profit made by company A 
\[ = \int_0^6 f(t) \, dt \]
\[ \approx \frac{1}{2} \left( f(0) + f(6) + 2\left(f(1) + f(2) + f(3) + f(4) + f(5)\right) \right) \]
\[ \approx 37.48705341 \]
\[ \approx 37.4871 \text{ billion dollars} \]

(ii) \( f(t) = \ln (e^t + 2) + 3 \)
\[ \frac{df(t)}{dt} = \frac{e^t}{e^t + 2} \]
\[ \frac{d^2f(t)}{dt^2} = \frac{(e^t + 2)e^t - e^t(e^t)}{(e^t + 2)^2} \]
\[ = \frac{2e^t}{(e^t + 2)^2} \]
Since \( \frac{d^2f(t)}{dt^2} > 0 \), \( f(t) \) is concave upward for \( 0 \leq t \leq 6 \).
Thus, the estimate in (a)(i) is an over-estimate.

(b) (i) \[ \frac{1}{40-t^2} = \frac{1}{40} \left(1 + \frac{t^2}{40} + \frac{t^4}{1600} + \ldots\right) = \frac{1}{40} + \frac{1}{1600}t^2 + \frac{1}{64000}t^4 + \ldots \]

(ii) Note that \( e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \ldots \). Hence, we have
\[ \frac{8e^t}{40-t^2} \]
\[ = 8\left( \frac{1}{40} + \frac{1}{1600}t^2 + \frac{1}{64000}t^4 + \ldots \right) \left(1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \ldots \right) \]
\[ = \frac{1}{5} + \frac{1}{5}t^2 + \frac{21}{200}t^3 + \frac{23}{600}t^4 + \frac{263}{2400}t^6 + \ldots \]

(iii) The total profit made by company B 
\[ = \int_0^6 g(t) \, dt \]
\[ \approx \int_0^6 \left( \frac{1}{5} + \frac{1}{10}t^2 + \frac{7}{200}t^3 + \frac{23}{2400}t^4 + \frac{263}{120000}t^6 \right) \, dt \]
\[ = 1.48224 \text{ billion dollars} \]

(c) Since the estimate in (b)(iii) is an under-estimate, we have
\[ \int_0^6 f(t) \, dt < 37.4871 < 1.48224 < \int_0^6 g(t) \, dt \]

Thus, Mary's claim is correct.

1A withhold 1A for omitting this step
1M for trapezoidal rule
1A \( a-1 \) for r.t. 37.487
1A
1M
1A f.t.
----------(7)
1A pp-1 for omitting \( `\ldots` \)
1M for any four terms correct
1A pp-1 for omitting \( `\ldots` \)
1M
1A for correct integration
1A \( a-1 \) for r.t. 41.822
----------(6)
1A
1A f.t.
----------(2)
9. (a) \[ A(t) = (-t^3 + 5t + a) e^{kt} + 7 \]

Since \( A(0) = 3 \), we have \( a + 7 = 3 \).

Thus, we have \( a = -4 \).

The required amount of water stored

\[ = (-1^3 + 5 - 4) e^k + 7 \]

\[ = 7 \text{ million cubic metres} \]

(b) \[ A(t) = (-t^3 + 5t - 4) e^{kt} + 7 \]

\[ \frac{dA(t)}{dt} = (-2t + 5) e^{kt} + (-t^3 + 5t - 4) (ke^{kt}) \]

\[ = \left(-kt^2 + (5k - 2)t + 5 - 4k\right) e^{kt} \]

Note that when \( t = 2 \), \( \frac{dA(t)}{dt} = 0 \).

So, we have \( 2k + 1 = 0 \).

Thus, we have \( k = -\frac{1}{2} \).

(c) (i) When \( A(t) \geq 7 \), we have

\[ -t^2 + 5t - 4 \geq 0 \]
\[ t^2 - 5t + 4 \leq 0 \]
\[ 1 \leq t \leq 4 \]

Thus, the adequate period lasts for 3 months.

(ii) Note that \( A(t) = (-t^3 + 5t - 4) e^{2t} + 7 \).

So, \[ \frac{dA(t)}{dt} = \left(\frac{t^2 - 9t}{2} + 7\right) e^{\frac{-t}{2}} \]

and \[ \frac{dA(t)}{dt} = 0 \] when \( t = 2 \) (rejected since \( t > 4 \)) or \( t = 7 \).

\[
\frac{dA(t)}{dt} = \begin{cases} 
< 0 & \text{if } 4 < t < 7 \\
= 0 & \text{if } t = 7 \\
> 0 & \text{if } 7 < t \leq 12 
\end{cases}
\]

So, \( A(t) \) attains its least value when \( t = 7 \).

The least amount of water stored

\[ = A(7) \approx 6.45647098 \]

\[ \approx 6.4564 \text{ million cubic metres} \]
Solution

Note that \( A(t) = (-t^2 + 5t - 4)e^{-t} + 7 \).

So, \( \frac{dA(t)}{dt} = \left( \frac{t^2 - 9t + 7}{2} \right)e^{-t} \)

and \( \frac{dA(t)}{dt} = 0 \) when \( t = 2 \) (rejected since \( t > 4 \)) or \( t = 7 \).

\[
\frac{d^2 A(t)}{dt^2} = \left( -\frac{t^2}{4} + \frac{9t}{4} - \frac{7}{2} \right)e^{-t} + \left( t - \frac{9}{2} \right)e^{-t} \\
= \left( -\frac{t^2}{4} + \frac{13t}{4} - 8 \right)e^{-t}
\]

Therefore, we have \( \left. \frac{d^2 A(t)}{dt^2} \right|_{t=7} = \frac{5}{2}e^{-7} > 0 \).

Note that there is only one local minimum after the adequate period.

So, \( A(t) \) attains its least value when \( t = 7 \).

The least amount of water stored
\( A(7) \approx 6.456447098 \)
\( = 6.4564 \text{ million cubic metres} \)

(iii) \( \frac{d^2 A(t)}{dt^2} = \left( -\frac{t^2}{4} + \frac{9t}{4} - \frac{7}{2} \right)e^{-t} + \left( t - \frac{9}{2} \right)e^{-t} \\
= \left( -\frac{t^2}{4} + \frac{13t}{4} - 8 \right)e^{-t} \\
\]

(iv) Since \( \frac{dA(t)}{dt} = \frac{1}{2} \left( t - \frac{9}{2} \right) \left( -\frac{25}{4} \right)e^{-t} > 0 \) for \( 10 < t < 12 \).

\( A(t) \) increases, within that year, after the adequate period has ended for 6 months.

Since \( \frac{d^2 A(t)}{dt^2} = \frac{1}{4} \left( t - \frac{13}{2} \right) \left( -\frac{41}{4} \right)e^{-t} < 0 \) for \( 10 < t < 12 \),

\( \frac{dA(t)}{dt} \) decreases, within that year, after the adequate period has ended for 6 months.

---

1A for \( t = 2 \) or 7
1A for testing + 1A
1A for \( \frac{d^2 A(t)}{dt^2} \)
1A for \( t=7 \)
1A a - 1 for t=6.456 million cubic metres
1A f.t.
10. (a) The required probability

\[ 1 - \left( \frac{2.4^0 e^{-2.4}}{0!} + \frac{2.4^1 e^{-2.4}}{1!} + \frac{2.4^2 e^{-2.4}}{2!} \right) \]

\[ \approx 0.430291254 \]

\[ \approx 0.4303 \]

(b) Let \( X \) be the expense of a customer. Then, \( X \sim N(375, 125^2) \).

The required probability

\[ P(300 < X < 600) \]

\[ = P\left( \frac{300 - 375}{125} < Z < \frac{600 - 375}{125} \right) \]

\[ = P(-0.6 < Z < 1.8) \]

\[ = 0.2257 + 0.4641 \]

\[ = 0.6898 \]

(c) The required probability

\[ = (0.25)(0.6898) + (0.8)(0.5 - 0.4641) \]

\[ = 0.20117 \]

\[ \approx 0.2012 \]

(d) The required probability

\[ \approx \frac{2.4^3 e^{-2.4}}{3!} \]

\[ \approx 0.00170163 \]

\[ \approx 0.0017 \]

(e) The required probability

\[ 0.00170163 + (0.20117)^3 \left( \frac{2.4^4 e^{-2.4}}{4!} \right) \]

\[ \approx 0.430291254 \]

\[ \approx 0.004431931 \]

\[ \approx 0.0044 \]

(f) Suppose that the revised least expense is \( x \).

Then, we have \( P(X \geq x) = 0.33 \).

So, we have \( P(Z \geq \frac{x - 375}{125}) = 0.33 \).

Therefore, we have \( \frac{x - 375}{125} = 0.44 \).

Hence, we have \( x = 430 \).

Thus, the revised least expense is $430.
11. Let \( X \) be the net weight of a can of brand \( D \) coffee beans. Then, \( X \sim N(300, 7.5^2) \).

(a) The required probability
\[
= P(X < 283.5 \text{ or } X > 316.5) \\
= P(Z < \frac{283.5 - 300}{7.5} \text{ or } Z > \frac{316.5 - 300}{7.5}) \\
= P(Z < -2.2 \text{ or } Z > 2.2) \\
= 2(0.0139) \\
= 0.0278
\]

(b) (i) The required probability
\[
= (1 - 0.0278)^1 (0.0278) \\
\approx 0.020387152 \\
\approx 0.0204
\]

(ii) The required probability
\[
= C_1^{30} (1 - 0.0278)^{29} (0.0278) \\
\approx 0.368195889 \\
\approx 0.3682
\]

(iii) The required probability
\[
= (1 - 0.0278)^{30} + 0.368195889 \\
\approx 0.797404575 \\
\approx 0.7974
\]

(c) (i) The required probability
\[
\approx \frac{1}{2} (0.368195889) \\
\approx 0.184097944 \\
\approx 0.1841
\]

(ii) The required probability
\[
\approx \frac{0.184097944}{0.797404575} \\
\approx 0.230871443 \\
\approx 0.2309
\]
12. (a) The required probability
\[
= \frac{4}{5} \left( \frac{1}{5} \right) + \left( \frac{4}{5} \right)^2 \left( \frac{1}{5} \right) + \left( \frac{4}{5} \right)^3 \left( \frac{1}{5} \right) + \ldots
\]
\[
= \frac{5124}{15625}
\]
\[
= 0.327936
\]
\[
\approx 0.3279
\]

(b) The required probability
\[
= \frac{4}{5} \left( \frac{1}{5} \right) + \left( \frac{4}{5} \right)^2 \left( \frac{1}{5} \right) + \left( \frac{4}{5} \right)^3 \left( \frac{1}{5} \right) + \ldots
\]
\[
= \frac{\frac{4}{5}}{1 - \left( \frac{4}{5} \right)^2}
\]
\[
= \frac{4}{9}
\]
\[
= 0.4444444444
\]
\[
\approx 0.4444
\]

(c) The required probability
\[
\frac{4}{9} - \frac{5124}{15625}
\]
\[
= \frac{4 \cdot 5124}{9 \cdot 15625}
\]
\[
= \frac{4096}{15625}
\]
\[
= 0.262144
\]
\[
\approx 0.2621
\]

The required probability
\[
= 1 - \frac{5124}{15625}
\]
\[
= \frac{15625 - 4 \cdot 5124}{9 \cdot 15625}
\]
\[
= \frac{4096}{15625}
\]
\[
= 0.262144
\]
\[
\approx 0.2621
\]
<table>
<thead>
<tr>
<th>Solution</th>
<th>Marks</th>
</tr>
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| (d) (i) The required probability
\[
\frac{1}{2}(\frac{3}{7}) + \frac{1}{2}(1) = \frac{5}{7} 
\approx 0.714285714 
\approx 0.7143
\] | 1M for either case 1A a\(\sim\) for r.t. 0.714 |
| The required probability
\[
1 - \left(\frac{1}{2}\right)^2 \frac{4}{7} = \frac{5}{7} 
\approx 0.714285714 
\approx 0.7143
\] | 1M for complementary probability 1A |
| (ii) The required probability
\[
1 - \frac{5}{7} = \frac{2}{7} 
\approx 0.285714286 
\approx 0.2857
\] | 1M for \(1 - (d)(i)\) 1A a\(\sim\) for r.t. 0.286 |
| The required probability
\[
\frac{4}{9}(\frac{2}{7}) = \frac{2}{7} 
\approx 0.285714286 
\approx 0.2857
\] | 1M for denominator = (2)(7) 1A a\(\sim\) for r.t. 0.286 |
| (iii) The required probability
\[
\frac{\frac{4}{9}(\frac{2}{7})}{\frac{4}{9}(\frac{2}{7}) + (1 - \frac{4}{9})(1 - \frac{2}{7})} = \frac{8}{33} 
\approx 0.2424242424 
\approx 0.242
\] | 1M for \(\frac{pq}{pq + (1 - p)(1 - q)}\) + 1M for \(\begin{cases} p = (b) \quad \text{or} \quad p = (d)(ii) \\ q = (d)(ii) \quad \text{or} \quad q = (b) \end{cases}\) 1A a\(\sim\) for r.t. 0.242 (7) |