SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. A and B are two events. Suppose that $\mathrm{P}(A) = 0.75$ and $\mathrm{P}(B) = 0.8$. Let $\mathrm{P}(A \cap B) = k$.

   (a) Express $\mathrm{P}(A \cup B)$ in terms of k.

   (b) (i) Prove that $0.55 \le k \le 1$.

   (ii) Let A' and B' be the complementary events of A and B respectively. Using (b)(i) and the fact that $A' \cup B'$ is the complementary event of $A \cap B$, or otherwise, prove that $\mathrm{P}(A' \cup B') \le 0.45$.

   (7 marks)

2. A researcher models the rate of change of the number of fish in a lake by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{6}{(e^{\frac{t}{4}} + e^{\frac{-t}{4}})^2} ,$$

   where N is the number in thousands of fish in the lake recorded yearly and $t\ (\ge 0)$ is the time measured in years from the start of the research. It is known that $N = 8$ when $t = 0$.

   (a) Prove that $\dfrac{\mathrm{d}N}{\mathrm{d}t} = \dfrac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}} + 1)^2}$. Using the substitution $u = e^{\frac{t}{2}} + 1$, or otherwise, express N in terms of t.

   (b) Estimate the number of fish in the lake after a very long time.

   (6 marks)

3. Figure 1 shows a container (without a lid) consisting of a thin hollow hemisphere of radius r cm joined to the bottom of a right circular cylindrical thin pipe of base radius r cm. It is known that the area of the outer surface of the container is $162\pi\,\text{cm}^2$.

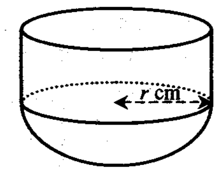


Figure 1

   (a) Prove that the capacity of the container is $(81\pi r - \dfrac{\pi r^3}{3})\ \text{cm}^3$.

   [Hint: The volume and the surface area of a sphere of radius r cm are $\dfrac{4}{3}\pi r^3\text{cm}^3$ and $4\pi r^2\text{cm}^2$ respectively, while the volume and the curved surface area of a right circular cylinder of base radius r cm and height h cm are $\pi r^2 h\,\text{cm}^3$ and $2\pi rh\,\text{cm}^2$ respectively.]

   (b) As r varies, can the capacity of the container be greater than $1600\ \text{cm}^3$? Explain your answer.

   (7 marks)

4. (a) (i) Expand $(x + y + z)^2$.

   (ii) Find the coefficients of x^3y , x^3z , xy^3 , y^3z , xz^3 and yz^3 in the expansion of $(x + y + z)^4$.

   (b) If a cup is randomly selected from a box containing red cups, blue cups and green cups, the probabilities of getting a red cup, a blue cup and a green cup are p , q and r respectively.

   If 4 cups are randomly selected from the box one by one with replacement, find, in terms of p , q and r ,

   (i) the probability that at least 2 cups of different colours are selected;

   (ii) the probability that exactly 3 cups of the same colour are selected.

   (7 marks)

Go on to the next page

5. Some statistics from a survey on the monthly incomes (in thousands of dollars) of a group of university graduates are summarized as follows:

Minimum	8
Maximum	52
Lower quartile	10
Median	17
Upper quartile	20
Mean	17.94
Standard deviation	4.7

(a) Using the above information, construct a box-and-whisker diagram to describe the distribution of the monthly incomes.

(b) A student proposes to model the distribution of the monthly incomes of the group of university graduates by a normal distribution with mean and standard deviation given in the above table.

(i) Using the model proposed by the student, find the probability that the monthly income of a randomly selected university graduate from the group is less than $ 17 000 .

(ii) Is the model proposed by the student appropriate? Explain your answer.

(6 marks)

6. David has forgotten his uncle's mobile phone number. He can only remember that the phone number is $98677XYZ$, where X , Y and Z are the forgotten digits. Find the probability that

(a) at least 2 of the forgotten digits are different;

(b) the forgotten digits are permutations of 2 , 3 and 8 ;

(c) exactly 2 of the forgotten digits have already appeared among the first five digits of the phone number.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C)2 answer book.

7. Define $f(x)=\dfrac{2x+5}{2x+6}$ and $g(x)=-f(x)$ for all $x \neq -3$.

Let C_1 and C_2 be the curves $y=f(x)$ and $y=g(x)$ respectively.

(a) (i) Find the equations of the vertical asymptote(s) and horizontal asymptote(s) to C_1 .

(ii) Sketch C_1 and indicate its asymptote(s) and its intercept(s).

(5 marks)

(b) On the diagram sketched in (a)(ii), sketch C_2 and indicate its asymptote(s), its intercept(s) and the point(s) of intersection of the two curves.

(4 marks)

(c) Let $\lambda > 7$. If the area enclosed by C_1 , C_2 and the straight line $x=\lambda$ is $(2\lambda+5+\ln(\lambda-7))$ square units, find the exact value(s) of λ .

(6 marks)

Go on to the next page

8. An engineer designed a driving test to compare fuel consumption when different driving tactics are used. The rates of change of fuel consumption in litres when using driving tactics A and B can be modelled respectively by

$$f(t) = \frac{1}{4} t (15 - t) e^{\frac{-t}{4}} \quad \text{and}$$

$$g(t) = \frac{1}{145} t (15 - t)^2 ,$$

where t (≥ 0) is the time measured in minutes from the start of the test.

(a) Use the trapezoidal rule with 5 sub-intervals to estimate the total fuel consumption from $t = 0$ to $t = 15$ when using driving tactic A.

(3 marks)

(b) Use integration to find the total fuel consumption from $t = 0$ to $t = 15$ when using driving tactic B.

(3 marks)

(c) Find the greatest value of $f(t)$, where $0 \leq t \leq 15$.

(5 marks)

(d) (i) Find $\frac{d^2 f(t)}{dt^2}$.

(ii) By considering $\frac{d^2 f(t)}{dt^2}$, can you determine whether the total fuel consumption from $t = 0$ to $t = 15$ when using driving tactic A will be less than that of using driving tactic B? Explain your answer.

(4 marks)

9. A researcher modelled the relationship between the atmospheric pressure y (in cm Hg) and the altitude x (in km) above sea-level by

$$\frac{dy}{dx} = -\alpha \beta^{-x} \qquad (x \geq 0),$$

where α and β are positive constants.

(a) It is known that $\ln(-\frac{dy}{dx})$ can be expressed as a linear function of x. The slope of the graph of the linear function is -0.125.

(i) Find the value of β correct to 3 decimal places.

(ii) The researcher found that the atmospheric pressures at sea-level (i.e. $x = 0$) and at an altitude of 2 km above sea-level were 76 cm Hg and 59.2 cm Hg respectively. If $\beta^{-x} = e^{-\lambda x}$ for all $x \geq 0$, find the value of λ. Hence or otherwise, find the value of α correct to 1 decimal place.

(8 marks)

(b) A balloon filled with helium gas is released from a point on a mountain. The altitude of the point is h km above sea-level. The balloon bursts when it reaches an altitude of $2h$ km above sea-level. The difference in the atmospheric pressures between the two altitudes is 13 cm Hg. It is also known that the atmospheric pressure at the top of the mountain is 25.2 cm Hg. Using the values of α and β obtained in (a),

(i) find the altitude of the mountain above sea-level correct to the nearest 0.1 km;

(ii) find the value(s) of h correct to 1 decimal place.

(7 marks)

Go on to the next page

10. A certain test gives a positive result in 94% of the people who have disease S. The test gives a positive result in 14% of the people who do not have disease S. In a city, 7.5% of the citizens have disease S.

(a) Find the probability that the test gives a positive result for a randomly selected citizen.

(3 marks)

(b) Given that the test gives a positive result for a randomly selected citizen, find the probability that the citizen does not have disease S.

(3 marks)

(c) The test is applied to a group of citizens one by one. Let M be the number of tests carried out when the first positive result is obtained. Denote the mean and the standard deviation of M by μ and σ respectively.

(i) Find $\mathrm{P}(M=3)$.

(ii) Find the exact values of μ and σ.

(iii) Using the fact that $\mathrm{P}(-k\sigma \le M-\mu \le k\sigma) \ge 1-\frac{1}{k^2}$ for any positive constant k, prove that $\mathrm{P}(1 \le M \le 25) \ge 0.95$.

(9 marks)

11. A manufacturer of brand C potato chips runs a promotion plan. Each packet of brand C potato chips contains either a red coupon or a blue coupon. Four red coupons can be exchanged for a toy. Five blue coupons can be exchanged for a lottery ticket. It is known that 30% of the packets contain red coupons and the rest contain blue coupons.

(a) Find the probability that a lottery ticket can be exchanged only when the 6th packet of brand C potato chips has been opened.

(3 marks)

(b) A person buys 10 packets of brand C potato chips.

(i) Find the probability that at least 1 toy can be exchanged.

(ii) Find the probability that exactly 1 toy and exactly 1 lottery ticket can be exchanged.

(iii) Given that at least 1 toy can be exchanged, find the probability that exactly 1 lottery ticket can also be exchanged.

(8 marks)

(c) Two persons buy 10 packets of brand C potato chips each. Assume that they do not share coupons or exchange coupons with each other.

(i) Find the probability that they can each get at least 1 toy.

(ii) Find the probability that one of them can get at least 1 toy and the other can get 2 lottery tickets.

(4 marks)

Go on to the next page

12. A customer who spends $\$ 300$ or more in a store during a visit is classified as a 'valuable' customer. The expenses of customers in the store are assumed to be independent and follow a normal distribution with a mean of $\$ 428$ and a standard deviation of $\$ 100$. The number of customers visiting the store in a minute can be modelled by a Poisson distribution with a mean of 4 customers per minute.

(a) Find the probability that a randomly selected customer of the store is a 'valuable' customer.

(2 marks)

(b) Find the probability that there are at least 2 customers visiting the store between 2:00 p.m. and 2:01 p.m. on a certain day.

(3 marks)

(c) Find the probability that there are exactly 3 customers visiting the store between 2:00 p.m. and 2:01 p.m. on a certain day and exactly 2 of them are 'valuable' customers.

(3 marks)

(d) Given that there are 2 or 3 customers visiting the store between 2:00 p.m. and 2:01 p.m. on a certain day, find the probability that exactly 2 of them are 'valuable' customers.

(3 marks)

(e) A customer who spends $\$ 600$ or more in the store during a visit will receive a gift. If the probability of the store giving out gifts is at least 0.99, find the smallest number of customers visiting the store.

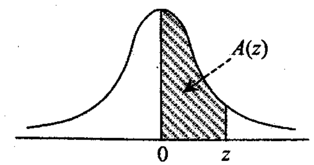
(4 marks)

END OF PAPER

Table: Area under the Standard Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

Note: An entry in the table is the proportion of the area under the entire curve which is between $z = 0$ and a positive value of z. Areas for negative values of z are obtained by symmetry.



$$A(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$$