Quadratic Equations – Part I

Before proceeding with this section we should note that the topic of solving quadratic equations will be covered in two sections. This is done for the benefit of those viewing the material on the web. This is a long topic and to keep page load times down to a minimum the material was split into two sections.

So, we are now going to solve quadratic equations. First, the **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

The only requirement here is that we have an x^2 in the equation. We guarantee that this term will be present in the equation by requiring $a \neq 0$. Note however, that it is okay if b and/or c are zero.

There are many ways to solve quadratic equations. We will look at four of them over the course of the next two sections. The first two methods won't always work, yet are probably a little simpler to use when they work. This section will cover these two methods. The last two methods will always work, but often require a little more work or attention to get correct. We will cover these methods in the next section.

So, let's get started.

Solving by Factoring

As the heading suggests we will be solving quadratic equations here by factoring them. To do this we will need the following fact.

$$\text{If } ab = 0 \text{ then either } a = 0 \text{ and/or } b = 0$$

This fact is called the **zero factor property** or **zero factor principle**. All the fact says is that if a product of two terms is zero then at least one of the terms had to be zero to start off with.

Notice that this fact will ONLY work if the product is equal to zero. Consider the following product.

$$ab = 6$$

In this case there is no reason to believe that either a or b will be 6. We could have $a = 2$ and $b = 3$ for instance. So, do not misuse this fact!

To solve a quadratic equation by factoring we first must move all the terms over to one side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form that can be factored. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don't have a zero on one side of the equation we won't be able to use the zero factor property.

Let's take a look at a couple of examples. Note that it is assumed that you can do the factoring at this point and so we won't be giving any details on the factoring. If you need a review of factoring you should go back and take a look at the Factoring section of the previous chapter.

Example 1 Solve each of the following equations by factoring.

(a) $x^2 - x = 12$ [Solution]

(b) $x^2 + 40 = -14x$ [Solution]

(c) $y^2 + 12y + 36 = 0$ [Solution]

(d) $4m^2 - 1 = 0$ [Solution]

(e) $3x^2 = 2x + 8$ [Solution]

(f) $10z^2 + 19z + 6 = 0$ [Solution]

(g) $5x^2 = 2x$ [Solution]

Solution

Now, as noted earlier, we won't be putting any detail into the factoring process, so make sure that you can do the factoring here.

(a) $x^2 - x = 12$

First, get everything on side of the equation and then factor.

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

Now at this point we've got a product of two terms that is equal to zero. This means that at least one of the following must be true.

$x - 4 = 0$	OR	$x + 3 = 0$
$x = 4$	OR	$x = -3$

Note that each of these is a linear equation that is easy enough to solve. What this tell us is that we have two solutions to the equation, $x = 4$ and $x = -3$. As with linear equations we can always check our solutions by plugging the solution back into the equation. We will check $x = -3$ and leave the other to you to check.

$$(-3)^2 - (-3) \stackrel{?}{=} 12$$

$$9 + 3 \stackrel{?}{=} 12$$

$$12 = 12 \qquad \text{OK}$$

So, this was in fact a solution.

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(b) $x^2 + 40 = -14x$

As with the first one we first get everything on side of the equal sign and then factor.

$$x^2 + 40 + 14x = 0$$

$$(x+4)(x+10) = 0$$

Now, we once again have a product of two terms that equals zero so we know that one or both of them have to be zero. So, technically we need to set each one equal to zero and solve. However, this is usually easy enough to do in our heads and so from now on we will be doing this solving in our head.

The solutions to this equation are,

$$x = -4 \qquad \text{AND} \qquad x = -10$$

To save space we won't be checking any more of the solutions here, but you should do so to make sure we didn't make any mistakes.

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(c) $y^2 + 12y + 36 = 0$

In this case we already have zero on one side and so we don't need to do any manipulation to the equation all that we need to do is factor. Also, don't get excited about the fact that we now have y's in the equation. We won't always be dealing with x's so don't expect to always see them.

So, let's factor this equation.

$$y^2 + 12y + 36 = 0$$
$$(y+6)^2 = 0$$
$$(y+6)(y+6) = 0$$

In this case we've got a perfect square. We broke up the square to denote that we really do have an application of the zero factor property. However, we usually don't do that. We usually will go straight to the answer from the squared part.

The solution to the equation in this case is,

$$y = -6$$

We only have a single value here as opposed to the two solutions we've been getting to this point. We will often call this solution a **double root** or say that it has **multiplicity of 2** because it came from a term that was squared.

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(d) $4m^2 - 1 = 0$

As always let's first factor the equation.

$$4m^2 - 1 = 0$$
$$(2m-1)(2m+1) = 0$$

Now apply the zero factor property. The zero factor property tells us that,

$$2m - 1 = 0 \qquad \text{OR} \qquad 2m + 1 = 0$$
$$2m = 1 \qquad \text{OR} \qquad 2m = -1$$
$$m = \frac{1}{2} \qquad \text{OR} \qquad m = -\frac{1}{2}$$

Again, we will typically solve these in our head, but we needed to do at least one in complete detail. So we have two solutions to the equation.

$$m = \frac{1}{2} \qquad \text{AND} \qquad m = -\frac{1}{2}$$

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(e) $3x^2 = 2x + 8$

Now that we've done quite a few of these, we won't be putting in as much detail for the next two problems. Here is the work for this equation.

$$3x^2 - 2x - 8 = 0$$

$$(3x+4)(x-2) = 0 \qquad \Rightarrow \qquad x = -\frac{4}{3} \quad \text{and} \quad x = 2$$

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(f) $10z^2 + 19z + 6 = 0$
Again, factor and use the zero factor property for this one.

$$10z^2 + 19z + 6 = 0$$

$$(5z+2)(2z+3) = 0 \qquad \Rightarrow \qquad z = -\frac{2}{5} \quad \text{and} \quad z = -\frac{3}{2}$$

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(g) $5x^2 = 2x$
This one always seems to cause trouble for students even though it's really not too bad.

First off. DO NOT CANCEL AN x FROM BOTH SIDES!!!! Do you get the idea that might be bad? It is. If you cancel an x from both sides, you WILL miss a solution so don't do it. Remember we are solving by factoring here so let's first get everything on one side of the equal sign.

$$5x^2 - 2x = 0$$

Now, notice that all we can do for factoring is to factor an x out of everything. Doing this gives,

$$x(5x-2) = 0$$

From the first factor we get that $x = 0$ and from the second we get that $x = \frac{2}{5}$. These are the two solutions to this equation. Note that is we'd canceled an x in the first step we would NOT have gotten $x = 0$ as an answer!

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Let's work another type of problem here. We saw some of these back in the Solving Linear Equations section and since they can also occur with quadratic equations we should go ahead and work on to make sure that we can do them here as well.

Example 2 Solve each of the following equations.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$ [Solution]

(b) $x + 3 + \dfrac{3}{x-1} = \dfrac{4-x}{x-1}$ [Solution]

Solution
Okay, just like with the linear equations the first thing that we're going to need to do here is to clear the denominators out by multiplying by the LCD. Recall that we will also need to note value(s) of x that will give division by zero so that we can make sure that these aren't included in the solution.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

The LCD for this problem is $(x+1)(2x-4)$ and we will need to avoid $x = -1$ and $x = 2$ to make sure we don't get division by zero. Here is the work for this equation.

$$\begin{aligned}(x+1)(2x-4)\left(\frac{1}{x+1}\right) &= (x+1)(2x-4)\left(1-\frac{5}{2x-4}\right)\\ 2x-4 &= (x+1)(2x-4)-5(x+1)\\ 2x-4 &= 2x^2-2x-4-5x-5\\ 0 &= 2x^2-9x-5\\ 0 &= (2x+1)(x-5)\end{aligned}$$

So, it looks like the two solutions to this equation are,

$$x = -\frac{1}{2} \qquad \text{and} \qquad x = 5$$

Notice as well that neither of these are the values of x that we needed to avoid and so both are solutions.

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(b) $x+3+\dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

In this case the LCD is $x-1$ and we will need to avoid $x=1$ so we don't get division by zero. Here is the work for this problem.

$$\begin{aligned}(x-1)\left(x+3+\frac{3}{x-1}\right) &= \left(\frac{4-x}{x-1}\right)(x-1)\\ (x-1)(x+3)+3 &= 4-x\\ x^2+2x-3+3 &= 4-x\\ x^2+3x-4 &= 0\\ (x-1)(x+4) &= 0\end{aligned}$$

So, the quadratic that we factored and solved has two solutions, $x=1$ and $x=-4$. However, when we found the LCD we also saw that we needed to avoid $x=1$ so we didn't get division by zero. Therefore, this equation has a single solution,

$$x = -4$$

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Before proceeding to the next topic we should address that this idea of factoring can be used to solve equations with degree larger than two as well. Consider the following example.

Example 3 Solve $5x^3 - 5x^2 - 10x = 0$.

Solution

The first thing to do is factor this equation as much as possible. In this case that means factoring out the greatest common factor first. Here is the factored form of this equation.

$$5x\left(x^2 - x - 2\right) = 0$$
$$5x\left(x - 2\right)\left(x + 1\right) = 0$$

Now, the zero factor property will still hold here. In this case we have a product of three terms that is zero. The only way this product can be zero is if one of the terms is zero. This means that,

$$5x = 0 \quad \Rightarrow \quad x = 0$$
$$x - 2 = 0 \quad \Rightarrow \quad x = 2$$
$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

So, we have three solutions to this equation.

So, provided we can factor a polynomial we can always use this as a solution technique. The problem is, of course, that it is sometimes not easy to do the factoring.

Square Root Property

The second method of solving quadratics we'll be looking at uses the **square root property**,

$$\text{If } p^2 = d \quad \text{then} \quad p = \pm\sqrt{d}$$

There is a (potentially) new symbol here that we should define first in case you haven't seen it yet. The symbol "$\pm$" is read as : "plus or minus" and that is exactly what it tells us. This symbol is shorthand that tells us that we really have two numbers here. One is $p = \sqrt{d}$ and the other is $p = -\sqrt{d}$. Get used to this notation as it will be used frequently in the next couple of sections as we discuss the remaining solution techniques. It will also arise in other sections of this chapter and even in other chapters.

This is a fairly simple property to use, however it can only be used on a small portion of the equations that we're ever likely to encounter. Let's see some examples of this property.

Example 4 Solve each of the following equations.

(a) $x^2 - 100 = 0$ [Solution]
(b) $25y^2 - 3 = 0$ [Solution]
(c) $4z^2 + 49 = 0$ [Solution]
(d) $\left(2t - 9\right)^2 = 5$ [Solution]
(e) $\left(3x + 10\right)^2 + 81 = 0$ [Solution]

Solution

There really isn't all that much to these problems. In order to use the square root property all that we need to do is get the squared quantity on the left side by itself with a coefficient of 1 and the number on the other side. Once this is done we can use the square root property.

(a) $x^2 - 100 = 0$

This is a fairly simple problem so here is the work for this equation.

$$x^2 = 100 \qquad\qquad x = \pm\sqrt{100} = \pm 10$$

So, there are two solutions to this equation, $x = \pm 10$. Remember this means that there are really two solutions here, $x = -10$ and $x = 10$.

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(b) $25y^2 - 3 = 0$

Okay, the main difference between this one and the previous one is the 25 in front of the squared term. The square root property wants a coefficient of one there. That's easy enough to deal with however; we'll just divide both sides by 25. Here is the work for this equation.

$$25y^2 = 3$$

$$y^2 = \frac{3}{25} \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{3}{25}} = \pm\frac{\sqrt{3}}{5}$$

In this case the solutions are a little messy, but many of these will do so don't worry about that. Also note that since we knew what the square root of 25 was we went ahead and split the square root of the fraction up as shown. Again, remember that there are really two solutions here, one positive and one negative.

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(c) $4z^2 + 49 = 0$

This one is nearly identical to the previous part with one difference that we'll see at the end of the example. Here is the work for this equation.

$$4z^2 = -49$$

$$z^2 = -\frac{49}{4} \qquad \Rightarrow \qquad z = \pm\sqrt{-\frac{49}{4}} = \pm i\sqrt{\frac{49}{4}} = \pm\frac{7}{2}i$$

So, there are two solutions to this equation : $z = \pm\frac{7}{2}i$. Notice as well that they are complex solutions. This will happen with the solution to many quadratic equations so make sure that you can deal with them.

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(d) $(2t - 9)^2 = 5$

This one looks different from the previous parts, however it works the same way. The square root property can be used anytime we have *something* squared equals a number. That is what we have here. The main difference of course is that the something that is squared isn't a single variable it is something else. So, here is the application of the square root property for this equation.

$$2t - 9 = \pm\sqrt{5}$$

Now, we just need to solve for *t* and despite the "plus or minus" in the equation it works the same way we would solve any linear equation. We will add 9 to both sides and then divide by a 2.

$$2t = 9 \pm \sqrt{5}$$

$$t = \frac{1}{2}\left(9 \pm \sqrt{5}\right) = \frac{9}{2} \pm \frac{\sqrt{5}}{2}$$

Note that we multiplied the fraction through the parenthesis for the final answer. We will usually do this in these problems. Also, do NOT convert these to decimals unless you are asked to. This is the standard form for these answers. With that being said we should convert them to decimals just to make sure that you can. Here are the decimal values of the two solutions.

$$t = \frac{9}{2} + \frac{\sqrt{5}}{2} = 5.61803 \qquad \text{and} \qquad t = \frac{9}{2} - \frac{\sqrt{5}}{2} = 3.38197$$

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(e) $\left(3x+10\right)^2 + 81 = 0$

In this final part we'll not put much in the way of details into the work.

$$\begin{aligned} \left(3x+10\right)^2 &= -81 \\ 3x+10 &= \pm 9\,i \\ 3x &= -10 \pm 9\,i \\ x &= -\frac{10}{3} \pm 3\,i \end{aligned}$$

So we got two complex solutions again and notice as well that with both of the previous part we put the "plus or minus" part last. This is usually the way these are written.

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As mentioned at the start of this section we are going to break this topic up into two sections for the benefit of those viewing this on the web. The next two methods of solving quadratic equations, completing the square and quadratic formula, are given in the next section.

Quadratic Equations – Part II

The topic of solving quadratic equations has been broken into two sections for the benefit of those viewing this on the web. As a single section the load time for the page would have been quite long. This is the second section on solving quadratic equations.

In the previous section we looked at using factoring and the square root property to solve quadratic equations. The problem is that both of these solution methods will not always work. Not every quadratic is factorable and not every quadratic is in the form required for the square root property.

It is now time to start looking into methods that will work for all quadratic equations. So, in this section we will look at completing the square and the quadratic formula for solving the quadratic equation,

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

Completing the Square

The first method we'll look at in this section is completing the square. It is called this because it uses a process called completing the square in the solution process. So, we should first define just what completing the square is.

Let's start with

$$x^2 + bx$$

and notice that the x^2 has a coefficient of one. That is required in order to do this. Now, to this lets add $\left(\frac{b}{2}\right)^2$. Doing this gives the following **factorable** quadratic equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

This process is called **completing the square** and if we do all the arithmetic correctly we can guarantee that the quadratic will factor as a perfect square.

Let's do a couple of examples for just completing the square before looking at how we use this to solve quadratic equations.

Example 1 Complete the square on each of the following.

(a) $x^2 - 16x$ [Solution]

(b) $y^2 + 7y$ [Solution]

Solution

(a) $x^2 - 16x$

Here's the number that we'll add to the equation.

$$\left(\frac{-16}{2}\right)^2 = (-8)^2 = 64$$

Notice that we kept the minus sign here even though it will always drop out after we square things. The reason for this will be apparent in a second. Let's now complete the square.

$$x^2 - 16x + 64 = (x - 8)^2$$

Now, this is a quadratic that hopefully you can factor fairly quickly. However notice that it will always factor as x plus the blue number we computed above that is in the parenthesis (in our case that is -8). This is the reason for leaving the minus sign. It makes sure that we don't make any mistakes in the factoring process.

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(b) $y^2 + 7y$

Here's the number we'll need this time.

$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

It's a fraction and that will happen fairly often with these so don't get excited about it. Also, leave it as a fraction. Don't convert to a decimal. Now complete the square.

$$y^2 + 7y + \frac{49}{4} = \left(y + \frac{7}{2}\right)^2$$

This one is not so easy to factor. However, if you again recall that this will ALWAYS factor as y plus the blue number above we don't have to worry about the factoring process.

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It's now time to see how we use completing the square to solve a quadratic equation. The process is best seen as we work an example so let's do that.

Example 2 Use completing the square to solve each of the following quadratic equations.

(a) $x^2 - 6x + 1 = 0$ [Solution]

(b) $2x^2 + 6x + 7 = 0$ [Solution]

(c) $3x^2 - 2x - 1 = 0$ [Solution]

Solution

We will do the first problem in detail explicitly giving each step. In the remaining problems we will just do the work without as much explanation.

(a) $x^2 - 6x + 1 = 0$

So, let's get started.

Step 1 : Divide the equation by the coefficient of the x^2 term. Recall that completing the square required a coefficient of one on this term and this will guarantee that we will get that. We don't need to do that for this equation however.

Step 2 : Set the equation up so that the x's are on the left side and the constant is on the right side.

$$x^2 - 6x = -1$$

Step 3 : Complete the square on the left side. However, this time we will need to add the number to both sides of the equal sign instead of just the left side. This is because we have to remember the rule that what we do to one side of an equation we need to do to the other side of the equation.

First, here is the number we add to both sides.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now, complete the square.

$$x^2 - 6x + 9 = -1 + 9$$
$$(x-3)^2 = 8$$

Step 4 : Now, at this point notice that we can use the square root property on this equation. That was the purpose of the first three steps. Doing this will give us the solution to the equation.

$$x - 3 = \pm\sqrt{8} \qquad \Rightarrow \qquad x = 3 \pm \sqrt{8}$$

And that is the process. Let's do the remaining parts now.

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(b) $2x^2 + 6x + 7 = 0$

We will not explicitly put in the steps this time nor will we put in a lot of explanation for this equation. This that being said, notice that we will have to do the first step this time. We don't have a coefficient of one on the x^2 term and so we will need to divide the equation by that first.

Here is the work for this equation.

$$x^2 + 3x + \frac{7}{2} = 0$$
$$x^2 + 3x = -\frac{7}{2} \qquad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$
$$x^2 + 3x + \frac{9}{4} = -\frac{7}{2} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = -\frac{5}{4}$$
$$x + \frac{3}{2} = \pm\sqrt{-\frac{5}{4}} \qquad \Rightarrow \qquad x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2} i$$

Don't forget to convert square roots of negative numbers to complex numbers!

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(c) $3x^2 - 2x - 1 = 0$

Again, we won't put a lot of explanation for this problem.

$$x^2 - \frac{2}{3}x - \frac{1}{3} = 0$$
$$x^2 - \frac{2}{3}x = \frac{1}{3}$$

At this point we should be careful about computing the number for completing the square since b is now a fraction for the first time.

$$\left(\frac{\frac{2}{3}}{2}\right)^2 = \left(\frac{2}{3}\cdot\frac{1}{2}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Now finish the problem.

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{3} + \frac{1}{9}$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{4}{9}$$

$$x - \frac{1}{3} = \pm\sqrt{\frac{4}{9}} \qquad \Rightarrow \qquad x = \frac{1}{3} \pm \frac{2}{3}$$

In this case notice that we can actually do the arithmetic here to get two integer and/or fractional solutions. We should always do this when there are only integers and/or fractions in our solution. Here are the two solutions.

$$x = \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1 \qquad \text{and} \qquad x = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

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A quick comment about the last equation that we solved in the previous example is in order. Since we received integer and factions as solutions, we could have just factored this equation from the start rather than used completing the square. In cases like this we could use either method and we will get the same result.

Now, the reality is that completing the square is a fairly long process and it's easy to make mistakes. So, we rarely actually use it to solve equations. That doesn't mean that it isn't important to know the process however. We will be using it in several sections in later chapters and is often used in other classes.

Quadratic Formula

This is the final method for solving quadratic equations and will always work. Not only that, but if you can remember the formula it's a fairly simple process as well.

We can derive the quadratic formula be completing the square on the general quadratic formula in standard form. Let's do that and we'll take it kind of slow to make sure all the steps are clear.

First, we MUST have the quadratic equation in standard form as already noted. Next, we need to divide both sides by a to get a coefficient of one on the x^2 term.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Next, move the constant to the right side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, we need to compute the number we'll need to complete the square. Again, this is one-half the coefficient of x, squared.

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Now, add this to both sides, complete the square and get common denominators on the right side to simplify things up a little.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now we can use the square root property on this.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Solve for x and we'll also simplify the square root a little.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

As a last step we will notice that we've got common denominators on the two terms and so we'll add them up. Doing this gives,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, summarizing up, provided that we start off in standard form,

$$ax^2 + bx + c = 0$$

and that's very important, then the solution to any quadratic equation is,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's work a couple of examples.

Example 3 Use the quadratic formula to solve each of the following equations.

(a) $x^2 + 2x = 7$ [Solution]

(b) $3q^2 + 11 = 5q$ [Solution]

(c) $7t^2 = 6 - 19t$ [Solution]

(d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$ [Solution]

(e) $16x - x^2 = 0$ [Solution]

Solution

The important part here is to make sure that before we start using the quadratic formula that we have the equation in standard form first.

(a) $x^2 + 2x = 7$

So, the first thing that we need to do here is to put the equation in standard form.

$$x^2 + 2x - 7 = 0$$

At this point we can identify the values for use in the quadratic formula. For this equation we have.

$$a = 1 \qquad b = 2 \qquad c = -7$$

Notice the "-" with c. It is important to make sure that we carry any minus signs along with the constants.

At this point there really isn't anything more to do other than plug into the formula.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{32}}{2} \end{aligned}$$

There are the two solutions for this equation. There is also some simplification that we can do. We need to be careful however. One of the larger mistakes at this point is to "cancel" to 2's in the numerator and denominator. Remember that in order to cancel anything from the numerator or denominator then it must be multiplied by the whole numerator or denominator. Since the 2 in the numerator isn't multiplied by the whole denominator it can't be canceled.

In order to do any simplification here we will first need to reduce the square root. At that point we can do some canceling.

$$x = \frac{-2 \pm \sqrt{(16)2}}{2} = \frac{-2 \pm 4\sqrt{2}}{2} = \frac{2\left(-1 \pm 2\sqrt{2}\right)}{2} = -1 \pm 2\sqrt{2}$$

That's a much nicer answer to deal with and so we will almost always do this kind of simplification when it can be done.

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(b) $3q^2 + 11 = 5q$

Now, in this case don't get excited about the fact that the variable isn't an x. Everything works the same regardless of the letter used for the variable. So, let's first get the equation into standard form.

$$3q^2 + 11 - 5q = 0$$

Now, this isn't quite in the typical standard form. However, we need to make a point here so that we don't make a very common mistake that many student make when first learning the quadratic formula.

Many students will just get everything on one side as we've done here and then get the values of a, b, and c based upon position. In other words, often students will just let a be the first number listed, b be the second number listed and then c be the final number listed. This is not correct however. For the quadratic formula a is the coefficient of the squared term, b is the coefficient of the term with just the variable in it (not squared) and c is the constant term. So, to avoid making this mistake we should always put the quadratic equation into the official standard form.

$$3q^2 - 5q + 11 = 0$$

Now we can identify the value of a, b, and c.

$$a = 3 \qquad b = -5 \qquad c = 11$$

Again, be careful with minus signs. They need to get carried along with the values.

Finally, plug into the quadratic formula to get the solution.

$$\begin{aligned} q &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(11)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25 - 132}}{6} \\ &= \frac{5 \pm \sqrt{-107}}{6} \\ &= \frac{5 \pm \sqrt{107}\, i}{6} \end{aligned}$$

As with all the other methods we've looked at for solving quadratic equations, don't forget to convert square roots of negative numbers into complex numbers. Also, when b is negative be very careful with the substitution. This is particularly true for the squared portion under the radical. Remember that when you square a negative number it will become positive. One of the more common mistakes here is to get in a hurry and forget to drop the minus sign after you square b, so be careful.

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(c) $7t^2 = 6 - 19t$

We won't put in quite the detail with this one that we've done for the first two. Here is the standard form of this equation.

$$7t^2 + 19t - 6 = 0$$

Here are the values for the quadratic formula as well as the quadratic formula itself.

$$a = 7 \qquad b = 19 \qquad c = -6$$

$$t = \frac{-19 \pm \sqrt{(19)^2 - 4(7)(-6)}}{2(7)}$$
$$= \frac{-19 \pm \sqrt{361 + 168}}{14}$$
$$= \frac{-19 \pm \sqrt{529}}{14}$$
$$= \frac{-19 \pm 23}{14}$$

Now, recall that when we get solutions like this we need to go the extra step and actually determine the integer and/or fractional solutions. In this case they are,

$$t = \frac{-19 + 23}{14} = \frac{2}{7} \qquad\qquad t = \frac{-19 - 23}{14} = -3$$

Now, as with completing the square, the fact that we got integer and/or fractional solutions means that we could have factored this quadratic equation as well.

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(d) $\frac{3}{y-2} = \frac{1}{y} + 1$

So, an equation with fractions in it. The first step then is to identify the LCD.

$$\text{LCD} : y(y-2)$$

So, it looks like we'll need to make sure that neither $y = 0$ or $y = 2$ is in our answers so that we don't get division by zero.

Multiply both sides by the LCD and then put the result in standard form.

$$(y)(y-2)\left(\frac{3}{y-2}\right) = \left(\frac{1}{y} + 1\right)(y)(y-2)$$
$$3y = y - 2 + y(y-2)$$
$$3y = y - 2 + y^2 - 2y$$
$$0 = y^2 - 4y - 2$$

Okay, it looks like we've got the following values for the quadratic formula.

$$a = 1 \qquad b = -4 \qquad c = -2$$

Plugging into the quadratic formula gives,

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{24}}{2}$$
$$= \frac{4 \pm 2\sqrt{6}}{2}$$
$$= 2 \pm \sqrt{6}$$

Note that both of these are going to be solutions since neither of them are the values that we need to avoid.

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(e) $16x - x^2 = 0$

We saw an equation similar to this in the previous section when we were looking at factoring equations and it would definitely be easier to solve this by factoring. However, we are going to use the quadratic formula anyway to make a couple of points.

First, let's rearrange the order a little bit just to make it look more like the standard form.

$$-x^2 + 16x = 0$$

Here are the constants for use in the quadratic formula.

$$a = -1 \qquad b = 16 \qquad c = 0$$

There are two things to note about these values. First, we've got a negative *a* for the first time. Not a big deal, but it is the first time we've seen one. Secondly, and more importantly, one of the values is zero. This is fine. It will happen on occasion and in fact, having one of the values zero will make the work much simpler.

Here is the quadratic formula for this equation.

$$x = \frac{-16 \pm \sqrt{(16)^2 - 4(-1)(0)}}{2(-1)}$$
$$= \frac{-16 \pm \sqrt{256}}{-2}$$
$$= \frac{-16 \pm 16}{-2}$$

Reducing these to integers/fractions gives,

$$x = \frac{-16+16}{-2} = \frac{0}{2} = 0 \qquad\qquad x = \frac{-16-16}{-2} = \frac{-32}{-2} = 16$$

So we get the two solutions, $x = 0$ and $x = 16$. These are exactly the solutions we would have gotten by factoring the equation.

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To this point in both this section and the previous section we have only looked at equations with integer coefficients. However, this doesn't have to be the case. We could have coefficient that are fractions or decimals. So, let's work a couple of examples so we can say that we've seen something like that as well.

Example 4 Solve each of the following equations.

(a) $\frac{1}{2}x^2 + x - \frac{1}{10} = 0$ [Solution]

(b) $0.04x^2 - 0.23x + 0.09 = 0$ [Solution]

Solution

(a) There are two ways to work this one. We can either leave the fractions in or multiply by the LCD (10 in this case) and solve that equation. Either way will give the same answer. We will only do the fractional case here since that is the point of this problem. You should try the other way to verify that you get the same solution.

In this case here are the values for the quadratic formula as well as the quadratic formula work for this equation.

$$a = \frac{1}{2} \qquad b = 1 \qquad c = -\frac{1}{10}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{10}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{1 + \frac{1}{5}}}{1} = -1 \pm \sqrt{\frac{6}{5}}$$

In these cases we usually go the extra step of eliminating the square root from the denominator so let's also do that,

$$x = -1 \pm \frac{\sqrt{6}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = -1 \pm \frac{\sqrt{(6)(5)}}{5} = -1 \pm \frac{\sqrt{30}}{5}$$

If you do clear the fractions out and run through the quadratic formula then you should get exactly the same result. For the practice you really should try that.

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(b) In this case do not get excited about the decimals. The quadratic formula works in exactly the same manner. Here are the values and the quadratic formula work for this problem.

$$a = 0.04 \qquad b = -0.23 \qquad c = 0.09$$

$$\begin{aligned} x &= \frac{-(-0.23) \pm \sqrt{(-0.23)^2 - 4(0.04)(0.09)}}{2(0.04)} \\ &= \frac{0.23 \pm \sqrt{0.0529 - 0.0144}}{0.08} \\ &= \frac{0.23 \pm \sqrt{0.0385}}{0.08} \end{aligned}$$

Now, to this will be the one difference between these problems and those with integer or fractional coefficients. When we have decimal coefficients we usually go ahead and figure the two individual numbers. So, let's do that,

$$x = \frac{0.23 \pm \sqrt{0.0385}}{0.08} = \frac{0.23 \pm 0.19621}{0.08}$$

$$x = \frac{0.23 + 0.19621}{0.08} \qquad \text{and} \qquad x = \frac{0.23 - 0.19621}{0.08}$$

$$= 5.327625 \qquad \text{and} \qquad = 0.422375$$

Notice that we did use some rounding on the square root.

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Over the course of the last two sections we've done quite a bit of solving. It is important that you understand most, if not all, of what we did in these sections as you will be asked to do this kind of work in some later sections.

Solving Quadratic Equations : A Summary

In the previous two sections we've talked quite a bit about solving quadratic equations. A logical question to ask at this point is which method should we use to solve a given quadratic equation? Unfortunately, the answer is, it depends.

If your instructor has specified the method to use then that, of course, is the method you should use. However, if your instructor had NOT specified the method to use then we will have to make the decision ourselves. Here is a general set of guidelines that *may* be helpful in determining which method to use.

1. Is it clearly a square root property problem? In other words, does the equation consist ONLY of something squared and a constant. If this is true then the square root property is probably the easiest method for use.
2. Does it factor? If so, that is probably the way to go. Note that you shouldn't spend a lot of time trying to determine if the quadratic equation factors. Look at the equation and if you can quickly determine that it factors then go with that. If you can't quickly determine that it factors then don't worry about it.
3. If you've reached this point then you've determined that the equation is not in the correct for the square root property and that it doesn't factor (or that you can't quickly see that it factors). So, at this point you're only real option is the quadratic formula.

Once you've solve enough quadratic equations the above set of guidelines will become almost second nature to you and you will find yourself going through them almost without thinking.

Notice as well that nowhere in the set of guidelines was completing the square mentioned. The reason for this is simply that it's a long method that is prone to mistakes when you get in a hurry. The quadratic formula will also always work and is much shorter of a method to use. In general, you should only use completing the square if your instructor has required you to use it.

As a solving technique completing the square should always be your last choice. This doesn't mean however that it isn't an important method. We will see the completing the square process arise in several sections in later chapters. Interestingly enough when we do see this process in later sections we won't be solving equations! This process is very useful in many situations of which solving is only one.

Before leaving this section we have one more topic to discuss. In the previous couple of sections we saw that solving a quadratic equation in standard form,

$$ax^2 + bx + c = 0$$

we will get one of the following three possible solution sets.

1. Two real distinct (*i.e.* not equal) solutions.
2. A double root. Recall this arises when we can factor the equation into a perfect square.
3. Two complex solutions.

These are the ONLY possibilities for solving quadratic equations in standard form. Note however, that if we start with rational expression in the equation we may get different solution sets because we may need avoid one of the possible solutions so we don't get division by zero errors.

Now, it turns out that all we need to do is look at the quadratic equation (in standard form of course) to determine which of the three cases that we'll get. To see how this works let's start off by recalling the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ in the quadratic formula is called the **discriminant**. It is the value of the discriminant that will determine which solution set we will get. Let's go through the cases one at a time.

1. Two real distinct solutions. We will get this solution set if $b^2 - 4ac > 0$. In this case we will be taking the square root of a positive number and so the square root will be a real number. Therefore the numerator in the quadratic formula will be $-b$ plus or minus a real number. This means that the numerator will be two different real numbers. Dividing either one by $2a$ won't change the fact that they are real, nor will it change the fact that they are different.

2. A double root. We will get this solution set if $b^2 - 4ac = 0$. Here we will be taking the square root of zero, which is zero. However, this means that the "plus or minus" part of the numerator will be zero and so the numerator in the quadratic formula will be $-b$. In other words, we will get a single real number out of the quadratic formula, which is what we get when we get a double root.

3. Two complex solutions. We will get this solution set if $b^2 - 4ac < 0$. If the discriminant is negative we will be taking the square root of negative numbers in the quadratic formula which means that we will get complex solutions. Also, we will get two since they have a "plus or minus" in front of the square root.

So, let's summarize up the results here.

1. If $b^2 - 4ac > 0$ then we will get two real solutions to the quadratic equation.
2. If $b^2 - 4ac = 0$ then we will get a double root to the quadratic equation.
3. If $b^2 - 4ac < 0$ then we will get two complex solutions to the quadratic equation.

Example 1 Using the discriminant determine which solution set we get for each of the following quadratic equations.

(a) $13x^2 + 1 = 5x$ [Solution]

(b) $6q^2 + 20q = 3$ [Solution]

(c) $49t^2 + 126t + 81 = 0$ [Solution]

Solution

All we need to do here is make sure the equation is in standard form, determine the value of a, b, and c, then plug them into the discriminant.

(a) $13x^2 + 1 = 5x$
First get the equation in standard form.

$$13x^2 - 5x + 1 = 0$$

We then have,

$$a = 13 \qquad b = -5 \qquad c = 1$$

Plugging into the discriminant gives,

$$b^2 - 4ac = (-5)^2 - 4(13)(1) = -27$$

The discriminant is negative and so we will have two complex solutions. For reference purposes the actual solutions are,

$$x = \frac{5 \pm 3\sqrt{3}\, i}{26}$$

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(b) $6q^2 + 20q = 3$
Again, we first need to get the equation in standard form.

$$6q^2 + 20q - 3 = 0$$

This gives,

$$a = 6 \qquad b = 20 \qquad c = -3$$

The discriminant is then,

$$b^2 - 4ac = (20)^2 - 4(6)(-3) = 472$$

The discriminant is positive we will get two real distinct solutions. Here they are,

$$x = \frac{-20 \pm \sqrt{472}}{12} = \frac{-10 \pm \sqrt{118}}{6}$$

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(c) $49t^2 + 126t + 81 = 0$
This equation is already in standard form so let's jump straight in.

$$a = 49 \qquad b = 126 \qquad c = 81$$

The discriminant is then,

$$b^2 - 4ac = (126)^2 - 4(49)(81) = 0$$

In this case we'll get a double root since the discriminant is zero. Here it is,

$$x = -\frac{9}{7}$$

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For practice you should verify the solutions in each of these examples.