

附加數學科

Mathematical Induction

*Do you know variety is
the most popular
question type of M.I.?*





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數學歸納法的題形變化和應用

HKCEE Syllabus 會考課程

Mathematical Induction and its simple applications.

數學歸納法及其簡易應用

- **Application to proof of inequalities is not required.**

不包括應用於不等式之證明



(A) 3 Standard steps for solving M.I. questions.

i. What is Mathematical Induction

Consider the following proposition P(n).

$$P(n): 1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1) \text{ for all natural numbers } n.$$

How to prove this pattern is true for all natural numbers n?

$$\text{When } n = 1, \quad \text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{1}{2}(1)(1+1) = 1$$

$$\text{When } n = 2, \quad \text{L.H.S.} = 1 + 2 = 3$$

$$\text{R.H.S.} = \frac{1}{2}(2)(2+1) = 3$$

$$\text{When } n = 3, \quad \text{L.H.S.} = 1 + 2 + 3 = 6$$

$$\text{R.H.S.} = \frac{1}{2}(3)(3+1) = 6$$

When $n = 4, 5, 6, \dots, 97, 98, 99,$

:

$$\text{When } n = 100, \quad \text{L.H.S.} = 1 + 2 + 3 + \dots + 100 = 5050$$

$$\text{R.H.S.} = \frac{1}{2}(100)(100+1) = 5050$$

How about $n = 1000, 10000, 100000, \dots$??? Infinite natural numbers !!!

Therefore, ...

Mathematical Induction 數學歸納法

When to Use : To prove a proposition P(n) is true for all natural numbers n.

Necessary Condition: P(n) can be proved if considering successive cases step by step.



ii. 3 Standard Steps for M.I. 數學歸納法三個基本步驟

Mathematical Induction 數學歸納法

Step 1: Prove $P(0/1/2)$ is true

Step 2: Assume $P(k)$ is true

Step 3: Consider $P(k+1)$, Prove $P(k+1)$ is true using $P(k)$ is true assumption.

e.g.1 Prove, by M.I., $P(n)$ is true for all natural numbers n .

$$P(n): 1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1) \text{ for all natural numbers } n.$$

Sol: When $n = 1$,

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{1}{2}(1)(1+1) = 1$$

$\therefore \text{L.H.S.} = \text{R.H.S.}$, $P(1)$ is true.

} Step 1

$$\text{Assume } P(k) \text{ is true, i.e. } 1 + 2 + 3 + \dots + k = \frac{1}{2}k(k+1)$$

} Step 2

where k is a positive integer.

Consider $P(k+1)$,

$$\text{L.H.S.} = 1 + 2 + 3 + \dots + k + k + 1$$

$$= \frac{1}{2}(k)(k+1) + (k+1) \text{ [By Assumption } P(k) \text{ is true]}$$

$$= \frac{1}{2}(k+1)(k+2)$$

$$= \frac{1}{2}(k+1)(k+2)$$

$$= \text{R.H.S.}$$

} Step 3

$\therefore P(k+1)$ is true, if $P(k)$ is true.

By M.I., $P(n)$ is true for all natural numbers n

p.s. Question Type: Simple Equality !!!



(B) Different Types of M.I. questions

Four Common Types of M.I. Questions

- a. Simple Equality
- b. Fraction Equality
- c. Divisible
- d. Derivation (Variety)

黎 Sir 提提你  :



Exam Type Questions:

a. Simple Equality 簡單等式

Skill 1: 無中生有大法

Skill 2: 望住最後一步

e.g.2 Prove, by M.I., $P(n)$ is true for all natural numbers n .

$$P(n): 1+2+3+\dots+n = \frac{1}{2}n(n+1) \text{ for all natural numbers } n.$$

黎 Sir 提提你 :

Sol: When 當 $n=1$,

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{1}{2}(1)(1+1) = 1$$

} Step 1

\therefore L.H.S. = R.H.S., $P(1)$ is true. $P(1)$ 成立.

Assume 假設 $P(k)$ is true, i.e. $1+2+3+\dots+k = \frac{1}{2}k(k+1)$

} Step 2

Where k is a positive integer.

Consider 考慮 $P(k+1)$,

$$\text{L.H.S.} = 1+2+3+\dots+k+k+1$$

$$= \frac{1}{2}(k)(k+1) + (k+1) \quad [\text{By Assumption } P(k) \text{ is true}]$$

$$= \frac{1}{2}(k+1)(k+2) \quad \text{無中生有大法!!!}$$

$$= \frac{1}{2}(k+1)(k+2) \quad \text{望住最後一步!!!}$$

$$= \text{R.H.S.}$$

} Step 3

\therefore $P(k+1)$ is true, if $P(k)$ is true.

By M.I., $P(n)$ is true for all natural numbers n



e.g.3 Prove, by M.I., P(n) is true for all natural numbers n.

$$P(n): 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2} \text{ for all natural numbers } n.$$

黎 Sir 提提你  :

When n=1,

$$\text{L.H.S.} = 1^2 = (-1)^{1-1} \frac{1(1+1)}{2} = \text{R.H.S.}$$

So P(1) is true.

$$\text{Assume } P(k) \text{ is true, i.e. } 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} k^2 = (-1)^{k-1} \frac{k(k+1)}{2},$$

where k is a positive integer.

Consider P(k+1),

$$\text{L.H.S.} = 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} k^2 + (-1)^{k+1-1} (k+1)^2$$

$$= (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^{k+1-1} (k+1)^2 \quad [\text{By Assumption } P(k) \text{ is true}]$$

$$= (-1)^k \frac{(k+1)}{2} [(-1)^{-1} k + 2(k+1)] \quad \text{無中生有大法!!!}$$

$$= (-1)^k \frac{(k+1)}{2} [(-1)k + 2k + 2] \quad (-1)^k = -1 \text{ if } k \text{ is odds !!!}$$

$$(-1)^k = 1 \text{ if } k \text{ is even !!!}$$

$$= (-1)^k \frac{(k+1)}{2} [(-1k + 2k + 2)]$$

$$= (-1)^k \frac{(k+1)}{2} [(k+2)]$$

$$= (-1)^k \frac{(k+1)(k+2)}{2}$$

$$= (-1)^{k+1-1} \frac{(k+1)(k+2)}{2} \quad \text{望住最後一步!!!}$$

$$= \text{R.H.S.}$$

∴ P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n

**b. Fraction Equality 份數等式****Skill 1:** 無中生有大法 + / 通份母**Skill 2:** 望住最後一步e.g.4 Prove, by M.I., $P(n)$ is true for all natural numbers n .

$$P(n): \frac{1 \times 2}{2 \times 3} + \frac{2 \times 2^2}{3 \times 4} + \frac{3 \times 2^3}{4 \times 5} + \dots + \frac{n \times 2^n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1 \quad \text{for all natural numbers } n.$$

黎 Sir 提提你 When $n=1$,

$$\text{L.H.S.} = \frac{1 \times 2}{2 \times 3} = \frac{1}{3}$$

$$\text{R.H.S.} = \frac{2^{n+1}}{n+2} - 1 = \frac{1}{3} = \text{L.H.S.}$$

So $P(1)$ is true.

$$\text{Assume } P(k) \text{ is true, i.e. } \frac{1 \times 2}{2 \times 3} + \frac{2 \times 2^2}{3 \times 4} + \frac{3 \times 2^3}{4 \times 5} + \dots + \frac{k \times 2^k}{(k+1)(k+2)} = \frac{2^{k+1}}{k+2} - 1,$$

where k is a positive integer.Consider $P(k+1)$,

$$\begin{aligned} \text{L.H.S.} &= \frac{1 \times 2}{2 \times 3} + \frac{2 \times 2^2}{3 \times 4} + \frac{3 \times 2^3}{4 \times 5} + \dots + \frac{k \times 2^k}{(k+1)(k+2)} + \frac{(k+1) \times 2^{k+1}}{(k+1+1)(k+1+2)} \\ &= \frac{2^{k+1}}{k+2} - 1 + \frac{(k+1) \times 2^{k+1}}{(k+2)(k+3)} \quad \text{[By Assumption } P(k) \text{ is true]} \\ &= \frac{2^{k+1}}{k+2} \left(1 + \frac{k+1}{k+3}\right) - 1 \quad \text{無中生有大法!!!} \\ &= \frac{2^{k+1}}{k+2} \left(\frac{2k+4}{k+3}\right) - 1 \quad \text{通份母!!!} \\ &= \frac{2^{k+1}}{k+2} \cdot \frac{2(k+2)}{k+3} - 1 \\ &= \frac{2^{k+2}}{k+3} - 1 \\ &= \frac{2^{(k+1)+1}}{(k+1)+2} - 1 \quad \text{望住最後一步!!!} \\ &= \text{R.H.S.} \end{aligned}$$

 $\therefore P(k+1)$ is true, if $P(k)$ is true.By M.I., $P(n)$ is true for all natural numbers n



e.g.5 Prove, by M.I., P(n) is true for all natural numbers n.

$$P(n): \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \quad \text{for all natural numbers } n.$$

黎 Sir 提提你 :

When $n=1$,

$$\text{L.H.S.} = \frac{1}{2^1} = \frac{1}{2}$$

$$\text{R.H.S.} = 2 - \frac{1+2}{2^1} = \frac{1}{2} \quad \text{L.H.S.}$$

So P(1) is true.

Assume P(k) is true, i.e. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$, where k is a positive integer.

Consider P(k+1),

$$\begin{aligned} \text{L.H.S.} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \quad \text{[By Assumption P(k) is true]} \\ &= 2 + \frac{-2(k+2) + (k+1)}{2^{k+1}} \quad \text{通份母!!!} \\ &= 2 + \frac{-2k - 4 + k + 1}{2^{k+1}} \\ &= 2 + \frac{-k - 3}{2^{k+1}} \\ &= 2 + \frac{-(k+3)}{2^{k+1}} \\ &= 2 - \frac{k+3}{2^{k+1}} \\ &= 2 - \frac{(k+1)+2}{2^{k+1}} \quad \text{望住最後一步!!!} \\ &= \text{R.H.S.} \end{aligned}$$

\therefore P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n



c. Divisible 整除性

Skill 1: Let 設 P(n) be $a = bn$

Skill 2: $x^{k+1} \Rightarrow x \cdot x^k$

Skill 3: 數字分拆

Skill 4: 抽公因式(Common Factor)

Skill 5: 望住最後一步

e.g.6 Prove, by M.I., $8^n - 3^n$ is divisible by 5 for all positive integer n.

黎 Sir 提提你 :

Let P(n) be the statement: $8^n - 3^n = 5M$ where M is a integer

When $n = 1$,

L.H.S. = $8^1 - 3^1 = 5 = 5(1) = \text{R.H.S.}$

Skill 1 - Let 設 P(n) be $a = bn$

\therefore P(1) is true.

Assume 假設 P(k) is true, i.e. $8^k - 3^k = 5Q$ where Q is a integer.

Consider P(k+1),

L.H.S. = $8^{k+1} - 3^{k+1}$

= $8 \cdot 8^k - 3 \cdot 3^k$ **Skill 2 - $x^{k+1} \Rightarrow x \cdot x^k$**

= $(5+3) \cdot 8^k - 3 \cdot 3^k$

= $5 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k$ **Skill 3 數字分拆**

= $5 \cdot 8^k + 3 \cdot (8^k - 3^k)$ **Skill 4 - Common Factor 抽公因式**

= $5 \cdot 8^k + 3 \cdot (5Q)$

= $5 \cdot (8^k + 3Q)$ **[By Assumption P(k) is true]**

= $5 \cdot (R)$ where $R = 8^k + 3Q$ is an integer

= $5 \cdot (R)$

= R.H.S. **Skill 5 - 望住最後一步**

\therefore P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n



e.g.7 Prove, by M.I., $9^n - 1$ is divisible by 8 for all positive integer n.

黎 Sir 提提你  :

Let P(n) be the statement: $9^n - 1 = 8M$ where M is a positive integer

When $n = 1$,

L.H.S. = $9^1 - 1 = 8(1) = \text{R.H.S.}$

Skill 1 - Let P(n) be $a = bn$

\therefore P(1) is true.

Assume P(k) is true, i.e. $9^k - 1 = 8Q$ where Q is a positive integer.

Consider P(k+1),

L.H.S. = $9^{k+1} - 1$

= $9^k \cdot 9 - 1$

Skill 2 - $x^{k+1} \Rightarrow x \cdot x^k$

= $9^k \cdot (8+1) - 1$

Skill 3 數字分拆

= $9^k \cdot 8 + 9^k - 1$

= $9^k \cdot 8 + 8Q$

[By Assumption P(k) is true]

= $8(9^k + Q)$

= $8 \cdot R$ where R is an integer

Skill 4 - Common Factor 抽公因式

= R.H.S.

Skill 5 - 望住最後一步

\therefore P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n

**d. Derivation (Variety) 引伸 (變化)****Skill 1: 數字分拆****Skill 2: 併項法****Skill 3: 斬頭切尾**e.g.8 a. Prove, by M.I., $P(n)$ is true for all natural numbers n . $P(n): 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ for all natural numbers n .b. Hence, evaluate $1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + 50(52)$.黎 Sir 提提你 :When $n=1$,

$$\text{L.H.S.} = 1 \times (1+1) = 1 \times 2 = 2 = \frac{1}{3}(1)(2)(3) = \text{R.H.S.}$$

 $\therefore P(1)$ is true.Assume $P(k)$ is true, i.e. $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$,where k is a positive integer.Consider $P(k+1)$,

$$\text{L.H.S.} = 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)(k+1+1)$$

$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+1+1) \quad \text{[By Assumption } P(k) \text{ is true]}$$

$$= \frac{1}{3}(k+1)(k+2)(k+3) \quad \text{無中生有大法!!!}$$

$$= \frac{1}{3}(k+1)(k+1+1)(k+1+2) \quad \text{望住最後一步!!!}$$

$$= \text{R.H.S.}$$

 $\therefore P(k+1)$ is true, if $P(k)$ is true.By M.I., $P(n)$ is true for all natural numbers n



$$\begin{aligned} \text{b. } & 1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + 50(52) && \xrightarrow{\text{Skill 1 數字分拆}} \\ & = 1 \times (1 + 2) + 2 \times (1 + 3) + 3 \times (1 + 4) + \dots + 50(1 + 51) \\ & = 1 \times 1 + 1 \times 2 + 2 \times 1 + 2 \times 3 + 3 \times 1 + 3 \times 4 + \dots + 50 \times 1 + 50 \times 51 \\ & = (1 \times 1 + 2 \times 1 + 3 \times 1 + \dots + 50 \times 1) + (1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + 50 \times 51) \\ & = \frac{(1 + 50)(50)}{2} + \frac{1}{3}(50)(51)(52) \quad (\text{By part a}) && \xrightarrow{\text{Skill 2 併項法}} \\ & = 45475 && \xrightarrow{\text{Sum of A.P. !!!}} \end{aligned}$$



e.g.9 (a) Prove, by mathematical induction, that

$$2(2) + 3(2^2) + 4(2^3) + \dots + (n+1)(2^n) = n(2^{n+1}) \text{ for all positive integers } n.$$

(b) Show that $1(2) + 2(2^2) + 3(2^3) + \dots + 98(2^{98}) = 97(2^{99}) + 2$.

黎 Sir 提提你  :

(a) Let $P(n)$ be the statement $2(2) + 3(2^2) + 4(2^3) + \dots + (n+1)(2^n) = n(2^{n+1})$
for all positive integers n .

When $n=1$, L.H.S. = $2(2) = 2(2^{1+1}) =$ R.H.S.

So $P(1)$ is true.

Assume $P(k)$ is true, i.e. $2(2) + 3(2^2) + 4(2^3) + \dots + (k+1)(2^k) = k(2^{k+1})$

Consider $P(k+1)$,

$$\begin{aligned} \text{L.H.S.} &= 2(2) + 3(2^2) + 4(2^3) + \dots + (k+1)(2^k) + (k+1+1)(2^{k+1}) \\ &= k(2^{k+1}) + (k+2)(2^{k+1}) \quad \text{[By Assumption } P(k) \text{ is true]} \\ &= (k+1) \left[\frac{k}{k+1}(2^{k+1}) + \frac{k+2}{k+1}(2^{k+1}) \right] \quad \text{無中生有大法!!!} \\ &= (k+1) \left[\frac{k(2^{k+1}) + (k+2)(2^{k+1})}{k+1} \right] \\ &= (k+1) \left[\frac{2(k+1)(2^{k+1})}{k+1} \right] \\ &= (k+1)[2(2^{k+1})] \\ &= (k+1)(2^{k+2}) \\ &= (k+1)(2^{k+1+1}) = \text{R.H.S.} \quad \text{望住最後一步!!!} \end{aligned}$$

$\therefore P(k+1)$ is true, if $P(k)$ is true.

By M.I., $P(n)$ is true for all natural numbers n



$$\begin{aligned} \text{(b)} \quad & 1(2) + 2(2^2) + 3(2^3) + \dots + 98(2^{98}) \\ & = (2-1)(2) + (3-1)(2^2) + (4-1)(2^3) + \dots + (99-1)(2^{98}) \longrightarrow \text{Skill 1: 數字分拆} \\ & = (2)(2) + (3)(2^2) + (4)(2^3) + \dots + (99)(2^{98}) - [(1)(2) + (1)(2^2) + (1)(2^3) + \dots + (1)(2^9)] \\ & = 98(2^{98+1}) - [(2) + (2^2) + (2^3) + \dots + (2^{98})] \quad \text{Skill 2 併項法 (Grouping Terms)} \\ & = 98(2^{99}) - \left[\frac{2(1-2^{98})}{1-2} \right] \longrightarrow \text{Sum of G.P. !!!} \\ & = 97(2^{99}) + 2^{99} + \left[\frac{2-2^{99}}{1} \right] \\ & = 97(2^{99}) + 2 \end{aligned}$$



e.g.10 (a) Prove, by mathematical induction, that

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \dots + \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

for all positive integers n .

(b) Evaluate

$$\frac{1}{9 \cdot 10 \cdot 11 \cdot 12} + \frac{1}{10 \cdot 11 \cdot 12 \cdot 13} + \frac{1}{11 \cdot 12 \cdot 13 \cdot 14} + \dots + \frac{1}{19(20)(21)(22)}.$$

黎 Sir 提提你  :

(a)



(b)



e.g.11 (a) Prove, by mathematical induction, that

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all positive integers } n.$$

(b) Evaluate $2^2 + 4^2 + 6^2 + \dots + 100^2$.

(c) Evaluate $141 \times 142 + 142 \times 143 + 143 \times 144 + \dots + 180 \times 181$.

黎 Sir 提提你 :

(a)



(b)

(c)



e.g.12 a) Prove, by M.I., $32^{2n-1} + 1$ is divisible by 11 for all positive integer n .

利用數學歸納法，對於所有正整數 n ，證明 $32^{2n-1} + 1$ 能被 11 整除。

b) Is $2^{985} + 1$ divisible by 11? Please support you answers with reasons.

黎 Sir 提提你  :

(a)



(b)



HKALE / HKCEE / GCSE / GCE / F.1 – F.7

黎 sir 教室

Pure Maths, Further Maths, Applied Maths, Maths & Statistics, Maths, Add. Maths, Physics, Economics.

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