Mathematics

數學科

Binomial Theorem

二項式定理

<u>Contents 目錄:</u>

- **Binomial Theorem** 二項式定理 **(1 lesson 1** 堂**)**

<u>HKCEE Syllabus 會考課程</u>

The Binomial Theorem for positive integral indices
正整指數之二項式定理

- **Excluding determination of the greatest term and relations between coefficients.**
  不包括係數間之關係及求最大項

(A) Definition of factorial relationship with nCr
階乘的定義和與 nCr 的關係

i. Definition of Factorial 階乘的定義:

$n! = n(n-1)(n-2)\ldots(3)(2)(1)$ **where n is a positive integer 正整數.**

$0! = 1$ **[Definition of Zero Factorial 零階乘的定義]**

e.g.

$10! = (10)(9)(8)\ldots(3)(2)(1)$

e.g.

$(n-1)! = (n-1)(n-2)(n-3)\ldots(3)(2)(1)$

e.g.

$$\frac{9!}{5!} = \frac{9\times8\times7\times\ldots\times3\times2\times1}{5\times4\times3\times2\times1}$$

$$= 9\times8\times7\times6$$

$$= 3024$$

e.g.

$$\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{(n-3)(n-2)(n-1)\ldots(3)(2)(1)}$$

$$= n(n-1)(n-2)$$

ii. Definition of nCr nCr 的定義:

$nCr = \dfrac{n!}{(n-r)!r!}$ **where n and r is positive and** $n \ge r$.

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iii. Common Properties of nCr:

1. $C_r^n = C_{n-r}^n$

2. $C_0^n = C_n^n = 1$

3. $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$ **or** $C_{r-1}^n + C_r^n = C_r^{n+1}$

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e.g. Prove $C_r^n = C_{n-r}^n$.

Sol: L.H.S. $= C_r^n$

$$= \frac{n!}{(n-r)!r!}$$

做緊咩?

$$= \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{[n-(n-r)]!(n-r)!}$$

$$= C_{n-r}^n = R.H.S.$$

So $C_r^n = C_{n-r}^n$

e.g. Prove $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$.

Sol: L.H.S. $= C_r^n + C_{r+1}^n$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{[n-(r+1)]!(r+1)!}$$

$$= \frac{n!(r+1)+n!(n-r)}{(n-r)!(r+1)!} = \frac{n! \bullet r + n! + n \bullet n! - n! \bullet r}{(n-r)!(r+1)!}$$

$$= \frac{n \bullet n! + n!}{(n+1-r-1)!(r+1)!} = \frac{n!(n+1)}{(n+1-r-1)!(r+1)!}$$

$$= \frac{(n+1)!}{[n+1-(r+1)]!(r+1)!} = C_{r+1}^{n+1} = R.H.S..$$

So $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$

iv. Pascal's Triangle 楊輝三角

nCr when										
	n=0					1				
	n=1				1		1			
	n=2			1		2		1		
	n=3		1		3		3		1	
	n=4	1		4		6		4		1
	……	C_0^4		C_1^4		C_2^4		C_3^4		C_4^4

……

(B) Explanation of Binomial Theorem 二項式定理詳解

i. Consider Expanding 考慮展開 $(a+b)^n = (a+b)(a+b)(a+b)......(a+b)$

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ii. Binomial Theorem 二項式定理

$$(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + ... + C_r^n a^{n-r}b^r + ... + C_{n-1}^n ab^{n-1} + C_n^n b^n$$

$(1+x)^n = 1 + C_1^n x + C_2^n x^2 + ... + C_r^n x^r + ... + C_{n-1}^n x^{n-1} + x^n$ **When** $a=1$ **and** $b=x$.

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e.g. Expand. $(4m-3y)^3$

Sol: $(4m-3y)^3$

$$=C_0^3(4m)^3(-3y)^0+C_1^3(4m)^2(-3y)^1+C_2^3(4m)^1(-3y)^2+C_3^3(4m)^0(-3y)^3$$

$$=1(64m^3)(1)+3(16m^2)(-3y)+3(4m)(9y^2)+1(1)(-27y^3)$$
$$=64m^3-144m^2y)+108my^2-27y^3$$

e.g. Expand $\left(x+\frac{1}{x}\right)^4$

Sol: $\left(x+\frac{1}{x}\right)^4$

$$=C_0^4(x)^4(\frac{1}{x})^0+C_1^4(x)^3(\frac{1}{x})^1+C_2^4(x)^2(\frac{1}{x})^2+C_3^4(x)^1(\frac{1}{x})^3+C_4^4(x)^0(\frac{1}{x})^4$$

$$=1(x)^4(1)+4(x)^3(\frac{1}{x})+6(x)^2(\frac{1}{x^2})+4(x)^1(\frac{1}{x^3})+1(x)^0(\frac{1}{x^4})$$

$$=x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$$

e.g. Expand $(1+2x)^3$

Sol: $(1+2x)^3$

$$=C_0^3(2x)^0+C_1^3(2x)^1+C_2^3(2x)^2+C_3^3(2x)^3$$

$$=1+3(2x)+3(4x^2)+1(8x^3)$$
$$=1+6x+12x^2+8x^3$$

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(C) General Term of Binomial Function 二項式函數的通項

i. General Term 通項

General Term 通項 $= C_r^n a^{n-r} b^r$

$$(a+b)^n = a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \dots\dots + C_r^n a^{n-r} b^r + \dots\dots + C_{n-1}^n a b^{n-1} + b^n$$

1st Term　2nd Term　3rd Term　r+1th Term　nth Term　n+1th Term

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e.g. Find the General Term 通項 of $(1+2x)^3$

Sol: General Term 通項 $= C_r^n a^{n-r} b^r \quad = C_r^3 1^{3-r} (2x) \quad = C_r^3 2^r x^r$

e.g. Find the General Term 通項 of $\left(x+\dfrac{1}{x}\right)^4$

Sol: General Term 通項 $= C_r^n a^{n-r} b^r \quad = C_r^4 (x)^{4-r} (\dfrac{1}{x})^r = C_r^4 (x)^{4-r-r} = C_r^4 (x)^{4-2r}$

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(D) Variation of Binomial Theorem Questions
二項式定理問題的變化

Exam Type Questions:

1. **Direct Expansion 直接展開**
2. **Finding General Term 找通項**
3. **Finding Coefficients 找係數**
4. **Variation of Summation of nCr nCr 之和及變化**
5. **Finding n 找 n**

1. Direct Expansion 直接展開

Skill 1: Binomial Theorem 技巧 1: 二項式定理

e.g. Expand. $(4m-3y)^3$ **e.g. Expand** $\left(x+\frac{1}{x}\right)^4$ **e.g. Expand** $(1+2x)^3$

Sol: Pls refer to p.5

2. Finding General Term 找通項

Skill 1: General Term (r+1th Term) $=C_r^n a^{n-r} b^r$

技巧 1: 通項 (r+1th 項) $=C_r^n a^{n-r} b^r$

e.g. Find the General Term of $(1+2x)^3$ **e.g. Find the General Term of** $\left(x+\frac{1}{x}\right)^4$

Sol: Pls refer to p.6

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3. Finding Coefficients 找係數

Skill 1: 技巧 1: General Term 通項 $=C_r^n a^{n-r} b^r$

e.g. Find the Coefficient of x^3 **in the expansion of** $(2+3x)^5$

Sol: when $r=3$**, The General Term** $=C_3^5 2^{5-3}(3x)^3$

$=10(4)(27x^3)$

$=108x^3$

So the Coefficient of x^3 **is 108.**

e.g. Find the Coefficient of x^2 **in the expansion of** $\left(x-\frac{2}{x}\right)^4$

Sol: General Term of $\left(x-\frac{2}{x}\right)^4 = C_r^4 x^{4-r}(-\frac{2}{x})^r$

$=C_r^4 x^{4-r-r}(-2)^r$

$=C_r^4(-2)^r x^{4-2r}$

When $4-2r=2$, $r=1$.

So the coefficient of x^2 **is** $=C_1^4(-2)^1=4(-2)=-8$

e.g. Determine and find (If any) whether the expansion of $\left(2x^2-\frac{1}{x}\right)^9$ **consist of constant term and** x^2 **term.**

Sol: General Term of $\left(2x^2-\frac{1}{x}\right)^9 = C_r^9(2x^2)^{9-r}(-\frac{1}{x})^r = \frac{C_r^9 2^{9-r} x^{18-3r}}{(-1)^r} = \frac{C_r^9 2^{9-r} x^{18-3r}}{(-1)^r}$

When $18-3r=0, r=6,$ **So the Constant Term is** $=\frac{C_6^9 2^{9-6}}{(-1)^6}=84\bullet 8=672$

When $18-3r=2, r=\frac{16}{3}\neq$ **integer, So there is no** x^2 **term.**

Skill 2: Product of Binomial 技巧 2: 二項式相乘

e.g. Given $(1+3x)^4(1-2x)^5=1+ax+bx^2+\ldots$**, find the values of the constant a and the coefficient of** x^2 **b.**

Sol: Given $(1+3x)^4(1-2x)^5=1+ax+bx^2+\ldots$

$[1+c_1^4(3x)+c_2^4(3x)^2+\ldots][1+C_1^5(-2x)+C_2^5(-2x)^2+\ldots]=1+ax+bx^2+\ldots$

$1+(3\bullet c_1^4-2\bullet C_1^5)x+(9\bullet C_2^4+4\bullet C_2^5+3\bullet C_1^4\bullet(-2)\bullet C_1^5)x^2+\ldots=1+ax+bx^2+\ldots$

By comparing Coefficients, $a=3\bullet c_1^4-2\bullet C_1^5$, $b=9\bullet C_2^4+4\bullet C_2^5-6\bullet C_1^4\bullet C_1^5$

So $a=3\bullet4-2\bullet5=2$, $b=9\bullet6+4\bullet10-6\bullet4\bullet5=-26$

So $a=2$, $b=-26$

Skill 3: Sum of Binomial 技巧 3: 二項式相加

e.g. Given $(1+x)^6+(1+2x)^6$**, find the coefficient of** x^2

Sol: The coefficient of $x^2=C_2^6+C_2^6(2)^2=15+15\bullet4=75$

Skill 4: 技巧 4: Trinomial 三項式

e.g. Given $(1+x+ax^2)^6=1+6x+k_1x^2+k_2x^3+\ldots$**, Express** k_1 **and** k_2 **in terms of a.**

Sol: $(1+x+ax^2)^6=1+6x+k_1x^2+k_2x^3+\ldots$

$1+C_1^6x(1+ax)+C_2^6x^2(1+ax)^2+C_3^6x^3(1+ax)^3+\ldots=1+6x+k_1x^2+k_2x^3+\ldots$

$1+C_1^6x+C_1^6ax^2+C_2^6x^2+2aC_2^6x^3+a^2C_2^6x^4+C_3^6x^3+\ldots=1+6x+k_1x^2+k_2x^3+\ldots$

$k_1=C_1^6+C_2^6=6+15=21$, $k_2=2a_2^6+C_3^6=2\bullet15+20=50$

So $k_1=21, k_2=50$

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4. Variation of Summation of nCr　nCr 之和及變化

Skill 1: Substitute x=a 技巧 1: 代 x=a

e.g. By Considering and expanding $(1+x)^5$ **, find** $C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5$ **.**

Sol: $(1+x)^5 = C_0^5 x^0 + C_1^5 x^1 + C_2^5 x^2 + C_3^5 x^3 + C_4^5 x^4 + C_5^5 x^5$

Put 代入 **x=1,** $(1+1)^5 = C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5$

$C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5$

Skill 2: Differentiate before Substitute x=a

技巧 2: 取導數後再代 x=a

e.g. By Considering and expanding $(1+x)^5$ **, find** $C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5$ **.**

Sol: $(1+x)^5 = C_0^5 x^0 + C_1^5 x^1 + C_2^5 x^2 + C_3^5 x^3 + C_4^5 x^4 + C_5^5 x^5$

Differentiate both side with respect to x 在兩邊同時間對 **x** 取導數

$5(1+x)^4 = C_1^5 + 2C_2^5 x^1 + 3C_3^5 x^2 + 4C_4^5 x^3 + 5C_5^5 x^4$

Put 代入 **x=1,** $5(1+1)^4 = C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5$

$C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5 = 80$

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5. Finding n 找 n

Skill 1: 技巧 1: General Term 通項 $= C_r^n a^{n-r} b^r$

e.g. By expanding of $(ax + \frac{1}{x^2})^n$ **in descending powers of x where a>0, if the 4th term of the expansion is independent of x and equals** $\frac{21}{2}$**, find the value of a and n.**

Sol: Gerneral Term $= C_r^n (ax)^{n-r} (\frac{1}{x^2})^r = C_r^n a^{n-r} x^{n-r-2r} = C_r^n a^{n-r} x^{n-3r}$

The 4th Term $= C_3^n a^{n-3} x^{n-3(3)} = C_3^n a^{n-3} x^{n-9} = \frac{21}{2}$

So $n-9=0$ => $n=9$ **and So** $C_3^9 a^{9-3} = \frac{21}{2}$ => $84a^6 = \frac{21}{2}$ => $a^6 = \frac{21}{2 \bullet 84}$

=> $a^6 = \frac{1}{8}$ => $a^2 = \frac{1}{2}$ => $a = \frac{1}{\sqrt{2}}$ **or** $a = -\frac{1}{\sqrt{2}}$ **(Rejected)**

Skill 2: 技巧 2: Sum of Binomials 二項式相加

e.g. Given $(1+x)^n + (1+2x)^n$ **and the coefficient of** x^2 **is 75. Find n.**

Sol: The coefficient of $x^2 = C_2^n + C_2^n (2)^2 = 5 \bullet C_2^n = 75$

$C_2^n = 15$

$\frac{n!}{(n-2)!2!} = 15$ => $n(n-1) = 15 \bullet 2$ => $n^2 - n - 30 = 0$

$(n-6)(n+5) = 0$ => $n = 6$ **or** $n = -5$ **(Rejected)**

So $n = 6$

Skill 3: Fraction Equality: Indices Law $x^a \bullet x^b = x^{a+b}$

技巧 3: 份數等式: 指數 $x^a \bullet x^b = x^{a+b}$

e.g. Given $(x - \frac{3}{x})^2 (1+2x)^n$ **and the constant term is 210. Find n.**

Sol: Expand $(x - \frac{3}{x})^2 (1+2x)^n$ => $[x^2 - 2x(\frac{3}{x}) + (\frac{3}{x})^2][1 + C_1^n (2x) + C_2^n (2x)^2 + ...]$

The constant term is: $-6 + 3^2 \bullet C_2^n \bullet 2^2 = 210$ => $-6 + 36 \bullet \frac{(n)(n-1)}{2} = 210$

=> $-6 + 18n^2 - 18n = 210$ => $18n^2 - 18n - 216 = 0$ => $n^2 - n - 12 = 0$

=> $(n-4)(n+3) = 0$ => $n = 4$ **or** $n = -3$ **(Rejected)**

So $n = 4$

Skill 4: 技巧 4: Sum of Coefficients 係數運算

e.g. In the expansion of $(1+x)^n$ **, if the coefficient of** x, x^2 **and** x^3 **are in Arithmetic Sequence 等比數列, find the value of n.**

Sol: $(1+x)^n = C_0^n x^0 + C_1^n x^1 + C_2^n x^2 + C_3^n x^3 + \ldots$

Coefficient of x, x^2 **and** x^3 **are** C_1^n, C_2^n **and** C_3^n **respectively**

$$C_2^n = \frac{C_1^n + C_3^n}{2} \Rightarrow$$

$$\frac{n!}{(n-2)!(2!)} = \frac{1}{2}[n + \frac{n!}{(n-3)!(3!)}] \Rightarrow \frac{n(n-1)}{(2)} = \frac{1}{2}[n + \frac{n(n-1)(n-2)}{(6)}]$$

$$\Rightarrow \quad n(n-1) = [\frac{6n + n(n^2 - 3n + 2)}{(6)}] \Rightarrow 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$$

$$\Rightarrow \quad n^3 - 9n^2 + 14n = 0 \Rightarrow n(n^2 - 9n + 14) = 0 \Rightarrow n(n-7)(n-2) = 0$$

=> $n = 0$ **(Rejected) or** $n = 2$ **(Rejected) or** $n = 7$

=> **n=7**

Skill 5: 技巧 5: Trinomial 三項式

e.g. Given the coefficient of x^2 **after the expansion of** $(1+x+x^2)^n$ **is 21. Find n.**

Sol: $(1+x+x^2)^n = [1+x(1+x)]^n = 1 + C_1^n x(1+x) + C_2^n x^2 (1+x)^2 + \ldots$

The coefficient of $x^2 = C_1^n + C_2^n = \dfrac{2n + n^2 - n}{2} = 21$

$n^2 + n - 42 = 0$

$(n-6)(n+7) = 0$ => $n = 6$ **or** $n = -7$ **(Rejected)**

So $n = 6$

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