2001-CE
A MATH

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2001

ADDITIONAL MATHEMATICS

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **FOUR** questions in Section B.
2. Write your answers in the answer book provided. For Section A, there is no need to start each question on a fresh page.
3. All working must be clearly shown.
4. Unless otherwise specified, numerical answers must be **exact**.
5. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.
6. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$
$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$	

* *

Section A (62 marks)
Answer **ALL** questions in this section.

1. Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2x+1}\right)$.

(3 marks)

2. Find $\int \frac{x}{\sqrt{3x^2+1}}\,\mathrm{d}x$.

(Hint : Let $u = 3x^2 + 1$.)

(4 marks)

3. Given a point $A(4, 0)$. $P(h, k)$ is a variable point on the circle $C: x^2 + y^2 = 4$. Let M be the mid-point of AP.

(a) Express the coordinates of M in terms of h and k.

(b) Find the equation of the locus of M.

(4 marks)

4. Find the constant term in the expansion of $(2x^3 + \frac{1}{x})^8$.

(4 marks)

5. Simplify the following complex numbers :

(a) $\frac{1+i}{1-i}$,

(b) $\frac{(1+i)^{16}}{(1-i)^{15}}$.

(5 marks)

6. (a) If $\sin x + \cos x = r\cos(x+\theta)$ for all values of x, where $r > 0$ and $-\pi < \theta \le \pi$, find the values of r and θ.

(b) Find the general solution of the equation

$$\sin x + \cos x = \sqrt{2} .$$

(5 marks)

7. $P(2, 0)$ is a point on the curve $x-(1+\sin y)^5=1$. Find the equation of the tangent to the curve at P.

(5 marks)

8. Let $\mathbf{a}$, $\mathbf{b}$ be two vectors such that $|\mathbf{a}|=4$, $|\mathbf{b}|=3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

(a) Find $\mathbf{a}\cdot\mathbf{b}$.

(b) Find the value of k if the vectors $(\mathbf{a}+k\mathbf{b})$ and $(\mathbf{a}-2\mathbf{b})$ are perpendicular to each other.

(6 marks)

9. Let $p=\dfrac{x^2+2x+8}{x-2}$ $\cdots\cdots(*)$,

where x is real. By expressing (*) in the form $ax^2+bx+c=0$, find the range of possible values of $\dfrac{x^2+2x+8}{x-2}$.

Hence find the range of possible values of $\left|\dfrac{x^2+2x+8}{x-2}\right|$.

(6 marks)

10. Two lines $L_1: x+y-5=0$ and $L_2: 2x-3y=0$ intersect at a point A. Find the equations of the two lines passing through A whose distances from the origin are equal to 2.

(6 marks)

11. Solve $\dfrac{y}{y-2} \le 2$.

Hence, or otherwise, solve $\dfrac{2^x}{2^x - 2} \le 2$.

(6 marks)

12. Prove, by mathematical induction, that

$$1\times2 + 2\times3 + 3\times4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n.

Hence evaluate

$$1\times3 + 2\times4 + 3\times5 + \cdots + 50\times52 .$$

(8 marks)

* *

Section B (48 marks)
Answer any **FOUR** questions in this section.
Each question carries 12 marks.

13. The line $y = mx + 2$, where $m \ne 0$, intersects the parabola $y^2 = x$ at two distinct points A and B.

(a) Show that $m < \dfrac{1}{8}$.

(3 marks)

(b) Find, in terms of m, the coordinates of the mid-point of chord AB.

(3 marks)

(c) Let L be the perpendicular bisector of chord AB. If L passes through the point (0, –3), show that $6m^3 + m^2 - 4m + 1 = 0$.

Hence find the equation(s) of L.

(6 marks)

14. (a)

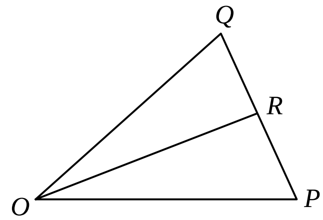


Figure 1(a)

In Figure 1 (a), OPQ is a triangle. R is a point on PQ such that $PR : RQ = r : s$.

Express $\overrightarrow{OR}$ in terms of r, s, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Hence show that if $\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$, then $m + n = 1$.

(3 marks)

(b)

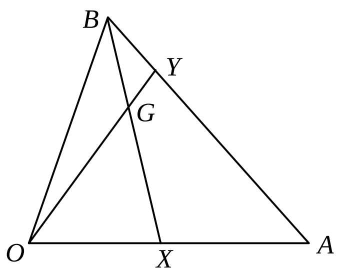


Figure 1(b)

In Figure 1 (b), OAB is a triangle. X is the mid-point of OA and Y is a point on AB. BX and OY intersect at point G where $BG : GX = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Using (a), express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(Hint : Put $\overrightarrow{OY} = k\overrightarrow{OG}$.)

(iii) Moreover, AG is produced to a point Z on OB. Let $\overrightarrow{OZ} = h\overrightarrow{OB}$.

(1) Find the value of h.

(2) Explain whether ZY is parallel to OA or not.

(9 marks)

15. (a)

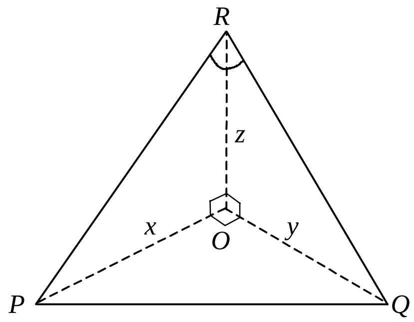


Figure 2 (a)

Figure 2 (a) shows a pyramid *OPQR*. The sides *OP*, *OQ* and *OR* are of lengths *x*, *y* and *z* respectively, and they are mutually perpendicular to each other.

(i) Express $\cos\angle PRQ$ in terms of x, y and z.

(ii) Let S_1, S_2, S_3 and S_4 denote the areas of ΔOPR, ΔOPQ, ΔOQR and ΔPQR respectively. Show that

$$S_4{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2 .$$

(6 marks)

(b)

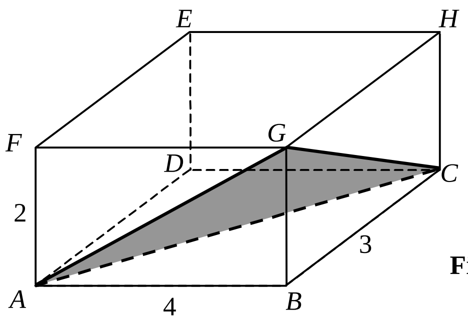


Figure 2 (b)

Figure 2 (b) shows a rectangular block *ABCDEFGH*. The lengths of sides *AB*, *BC* and *AF* are 4, 3 and 2 respectively. A pyramid *ABCG* is cut from the block along the plane *GAC*.

(i) Find the volume of the pyramid *ABCG*.

(ii) Find the angle between the side *AB* and the plane *GAC*, giving your answer correct to the nearest degree.

(6 marks)

16.

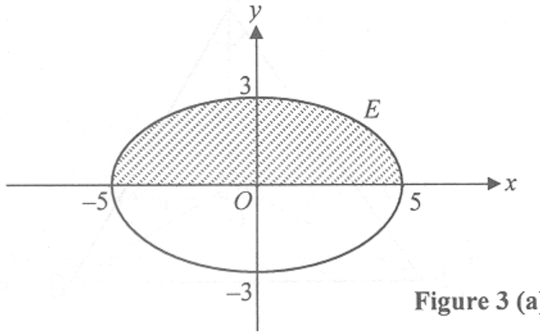


Figure 3 (a)

Figure 3 (a) shows the ellipse $E : \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$. The shaded region in the first and second quadrants is bounded by E and the x-axis.

(a) Using integration and the substitution $x = 5\sin\theta$, find the area of the shaded region.

(5 marks)

(b) A piece of chocolate is in the shape of the solid of revolution formed by revolving the shaded region in Figure 3 (a) about the x-axis. Using integration, find the volume of the chocolate.

(4 marks)

(c)

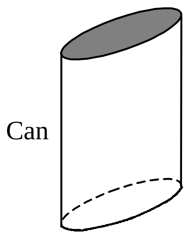


Figure 3 (b)

Four pieces of the chocolate mentioned in (b) are packed in a can, which is in the shape of a right non-circular cylinder (see Figure 3 (b)). The chocolates are placed one above the other. Their axes of revolution are parallel to the base of the can and in the same vertical plane. For economic reasons, the can is in a shape such that it can just hold the chocolates. Find the capacity of the can.

(3 marks)

17. P is a point in an Argand diagram representing the complex number z such that $|z-(\sqrt{3}+i)|=\sqrt{3}$. In Figure 4, the circle C is the locus of P.

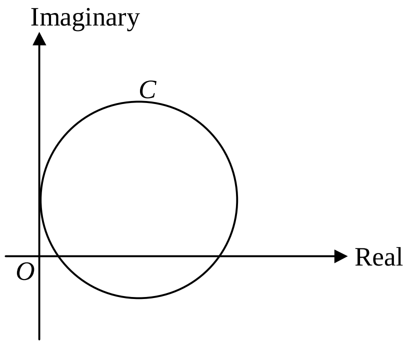


Figure 4

(a) Write down the coordinates of the centre and the radius of C.

(2 marks)

(b) Q is the point on C such that the modulus of the complex number represented by Q is the greatest. Find the complex number represented by Q.

(5 marks)

(c) R is the point on C such that the principal value of the argument of the complex number represented by R is the least. Find the complex number represented by R.

(5 marks)

18. Let $\mathrm{f}(x)$ be a polynomial, where $-2 \le x \le 10$. Figure 5 (a) shows a sketch of the curve $y = \mathrm{f}'(x)$, where $\mathrm{f}'(x)$ denotes the first derivative of $\mathrm{f}(x)$.

(a) (i) Write down the range of values of x for which $\mathrm{f}(x)$ is increasing.

(ii) Find the x-coordinates of the maximum and minimum points of the curve $y = \mathrm{f}(x)$.

(iii) In Figure 5 (b), draw a possible sketch of the curve $y = \mathrm{f}(x)$.

(6 marks)

(b) In Figure 5 (c), sketch the curve $y = \mathrm{f}''(x)$.

(2 marks)

(c) Let $\mathrm{g}(x) = \mathrm{f}(x) + x$, where $-2 \le x \le 10$.

(i) In Figure 5 (a), sketch the curve $y = \mathrm{g}'(x)$.

(ii) A student makes the following note :

Since the functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are different, the graphs of $y = \mathrm{f}''(x)$ and $y = \mathrm{g}''(x)$ should be different.

Explain whether the student is correct or not.

(4 marks)

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

If you attempt Question 18, fill in the first three boxes above and tie this sheet into your answer book.

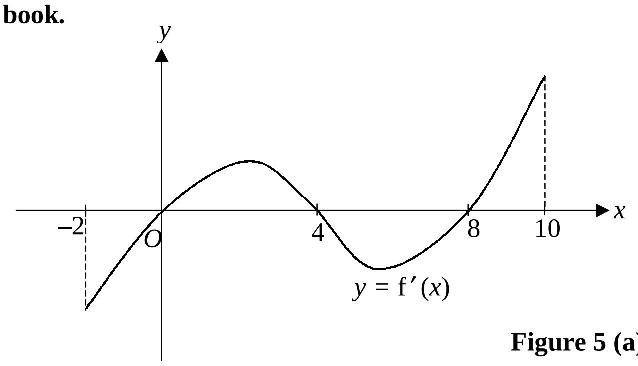


Figure 5 (a)

(a) (iii) **(continued)**

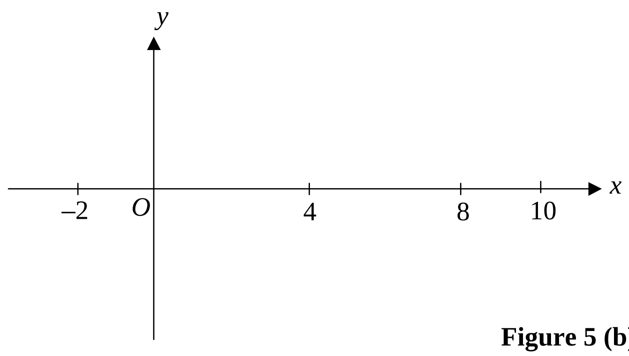


Figure 5 (b)

(b) **(continued)**

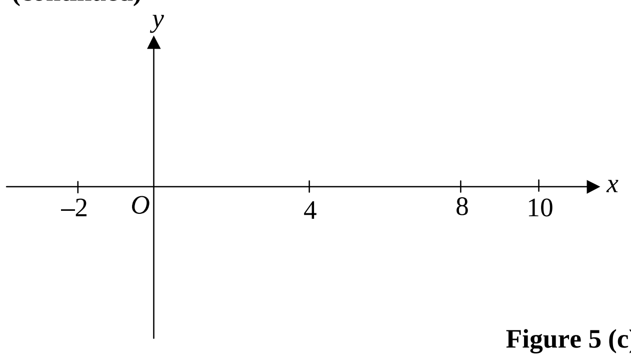


Figure 5 (c)

END OF PAPER

2001

Additional Mathematics

Section A

1. $\dfrac{2x(x+1)}{(2x+1)^2}$

2. $\dfrac{1}{3}(3x^2+1)^{\frac{1}{2}}+c$, where c is a constant

3. (a) $\left(\dfrac{h+4}{2},\dfrac{k}{2}\right)$

   (b) $x^2+y^2-4x+3=0$

4. 112

5. (a) i

   (b) $1-i$

6. (a) $\sqrt{2}$, $-\dfrac{\pi}{4}$

   (b) $x=2n\pi+\dfrac{\pi}{4}$, where n is an integer

7. $x-5y-2=0$

8. (a) 6

   (b) $\dfrac{1}{3}$

9. $\frac{x^2+2x+8}{x-2} \geq 14$ or $\frac{x^2+2x+8}{x-2} \leq -2$

$\left|\frac{x^2+2x+8}{x-2}\right| \geq 2$

10. $12x-5y-26=0$, $y-2=0$

11. $y \geq 4$ or $y < 2$
$x \geq 2$ or $x < 1$

12. 45475

Section B

Q.13 (a) $\begin{cases} y = mx + 2 \\ y^2 = x \end{cases}$

$(mx+2)^2 = x$

$m^2x^2 + (4m-1)x + 4 = 0$ - - - - - (*)

Since the line intersects the parabola at two distinct points,

discriminant $\Delta = (4m-1)^2 - 4m^2(4) > 0$

$16m^2 - 8m + 1 - 16m^2 > 0$

$m < \dfrac{1}{8}$

(b) Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively.

From (*), $x_1 + x_2 = \dfrac{-(4m-1)}{m^2}$

$\therefore$ x coordinate of mid-point $= \dfrac{x_1 + x_2}{2} = \dfrac{1-4m}{2m^2}$

y coordinate of mid-point $= mx + 2$

$= m(\dfrac{1-4m}{2m^2}) + 2 = \dfrac{1}{2m}$

$\therefore$ the coordinates of the mid-point are $(\dfrac{1-4m}{2m^2}, \dfrac{1}{2m})$.

(c) Slope of the perpendicular bisector of $AB = -\dfrac{1}{m}$

$$\frac{-3 - \frac{1}{2m}}{0 - \frac{1-4m}{2m^2}} = -\frac{1}{m}$$

$$-3 - \frac{1}{2m} = \frac{1}{m}(\frac{1-4m}{2m^2})$$

$$-3(2m^3) - m^2 = 1 - 4m$$

$$6m^3 + m^2 - 4m + 1 = 0$$

$(m+1)(6m^2-5m+1)=0$

$(m+1)(3m-1)(2m-1)=0$

$m=-1, \frac{1}{3}$ or $\frac{1}{2}$.

From (a), $m<\frac{1}{8}$ $\therefore m=-1$

The equation of L is

$y-\frac{1}{2(-1)}=\frac{-1}{(-1)}[x-\frac{1-4(-1)}{2(-1)^2}]$

$y=x-3$.

Q.14 (a) $\overrightarrow{OR} = \dfrac{s\overrightarrow{OP} + r\overrightarrow{OQ}}{r+s}$

Comparing coefficients with the expression

$\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$,

$m = \dfrac{s}{r+s}$ and $n = \dfrac{r}{r+s}$

$$m + n = \frac{s}{r+s} + \frac{r}{r+s}$$
$$= 1$$

(b) (i) $\overrightarrow{OG} = \dfrac{\frac{1}{2}\vec{a} + 3\vec{b}}{4} = \dfrac{1}{8}\vec{a} + \dfrac{3}{4}\vec{b}$

(ii) $\overrightarrow{OY} = k\overrightarrow{OG}$

$$= \frac{k}{8}\vec{a} + \frac{3k}{4}\vec{b}$$

Using (a), $\dfrac{k}{8} + \dfrac{3k}{4} = 1$

$$k = \frac{8}{7}$$

$$\therefore \overrightarrow{OY} = \frac{1}{8}(\frac{8}{7})\vec{a} + \frac{3}{4}(\frac{8}{7})\vec{b}$$
$$= \frac{1}{7}\vec{a} + \frac{6}{7}\vec{b}$$

(iii) (1) From (b) (i), $\overrightarrow{OG} = \dfrac{1}{8}\vec{a} + \dfrac{3}{4}\vec{b}$

$$= \frac{1}{8}\vec{a} + \frac{3}{4h}\overrightarrow{OZ}$$

Using (a), $\dfrac{1}{8} + \dfrac{3}{4h} = 1$

$$h = \frac{6}{7}$$

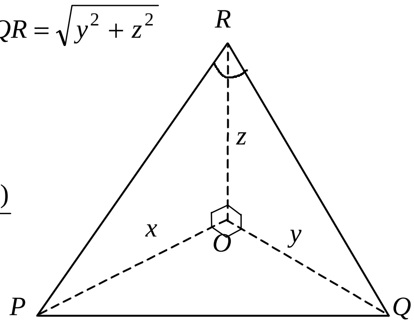
(2) $\overrightarrow{ZY} = \overrightarrow{OY} - \overrightarrow{OZ}$

$$= \frac{1}{7}\vec{a} + \frac{6}{7}\vec{b} - \frac{6}{7}\vec{b}$$

$$= \frac{1}{7}\vec{a}$$

As $\overrightarrow{ZY} = \lambda\overrightarrow{OA}$, so ZY is parallel to OA.

Q.15 (a) (i) $RP=\sqrt{x^2+z^2}$, $PQ=\sqrt{x^2+y^2}$, $QR=\sqrt{y^2+z^2}$



$$\cos\angle PRQ=\frac{RP^2+QR^2-PQ^2}{2(RP)(QR)}$$
$$=\frac{(x^2+z^2)+(y^2+z^2)-(x^2+y^2)}{2\sqrt{x^2+z^2}\sqrt{y^2+z^2}}$$
$$=\frac{z^2}{\sqrt{x^2+z^2}\sqrt{y^2+z^2}}$$

(ii) $S_1=\frac{1}{2}xz,\ S_2=\frac{1}{2}xy,\ S_3=\frac{1}{2}yz$

$$\sin\angle PRQ=\frac{\sqrt{(x^2+z^2)(y^2+z^2)-z^4}}{\sqrt{x^2+z^2}\sqrt{y^2+z^2}}$$

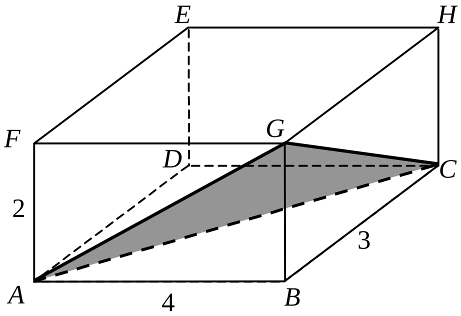
$$S_4=\frac{1}{2}(QR)(RP)\sin\angle PRQ$$
$$=\frac{1}{2}\sqrt{y^2+z^2}\sqrt{x^2+z^2}\ \frac{\sqrt{(x^2+z^2)(y^2+z^2)-z^4}}{\sqrt{x^2+z^2}\sqrt{y^2+z^2}}$$
$$=\frac{1}{2}\sqrt{(x^2+z^2)(y^2+z^2)-z^4}$$
$$S_4{}^2=\frac{1}{4}(x^2y^2+y^2z^2+z^2x^2)$$
$$=(\frac{xy}{2})^2+(\frac{yz}{2})^2+(\frac{zx}{2})^2$$
$$=S_1{}^2+S_2{}^2+S_3{}^2$$

(b) (i) Volume of $ABCG$

$=\frac{1}{3}\times$ base area $\times$ height

$=\frac{1}{3}\times(\frac{1}{2}\times4\times3)\times2$

$=4$



(ii) Using (a), area of $\Delta GAC = \sqrt{(\frac{4\times 2}{2})^2 + (\frac{2\times 3}{2})^2 + (\frac{4\times 3}{2})^2}$

$= \sqrt{61}$

Let h be the perpendicular distance from B to plane GAC.
Considering the volume of $ABCG$,

$$\frac{1}{3}(\sqrt{61})h = 4$$

$$h = \frac{12}{\sqrt{61}}$$

Let θ be the angle between AB and plane GAC.

$$\sin\theta = \frac{h}{AB}$$

$$= \frac{12/\sqrt{61}}{4}$$

$\theta = 23°$ (correct to the nearest degree)

Q.16 (a) Area $= \int_{-5}^{5} y\,\mathrm{d}x$

$$= \int_{-5}^{5} 3\sqrt{1-\frac{x^2}{25}}\,\mathrm{d}x$$

Put $x = 5\sin\theta$.

$\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$

$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\sqrt{1-\frac{25\sin^2\theta}{25}}(5\cos\theta\,\mathrm{d}\theta)$$

$$= 15\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

$$= 15\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\mathrm{d}\theta$$

$$= \frac{15}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{15}{2}\left[\frac{\pi}{2}+\frac{1}{2}\sin\pi+\frac{\pi}{2}-\frac{1}{2}\sin(-\pi)\right]$$

$$= \frac{15\pi}{2}$$

(b) Volume $= \int_{-5}^{5} \pi y^2 \, \mathrm{d}x$

$$= \pi \int_{-5}^{5} 9(1-\frac{x^2}{25})\mathrm{d}x$$

$$= 9\pi \left[x - \frac{x^3}{75} \right]_{-5}^{5}$$

$$= 9\pi \left[5 - \frac{125}{75} + 5 - \frac{125}{75} \right]$$

$$= 60\pi$$

(c) Capacity of the can

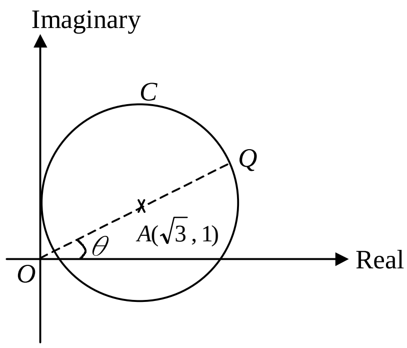
= Base area $\times$ height

$$= (\frac{15\pi}{2} \times 2) \times (6 \times 4)$$
$$= 360\pi$$

Q.17 (a) The centre is $(\sqrt{3}, 1)$

The radius is $\sqrt{3}$.

(b) The position of Q is shown below :



Let A denote the centre of C and θ be the angle between OA and the real axis.

$$OA = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$OQ = OA + AQ$$

$$= 2 + \sqrt{3}$$

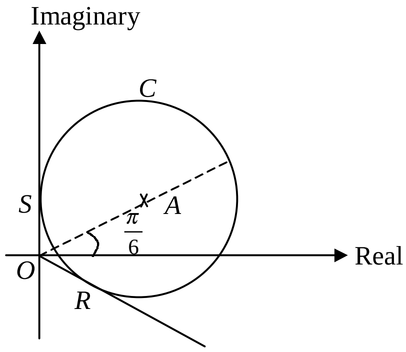
$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$\therefore$ the complex number represented by Q is

$$(2+\sqrt{3})\,(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}).$$

(c) The position of R is shown below (OR is a tangent to C) :



Let S be the point of contact between C and the imaginary axis.

$OR = OS = 1$

$$\angle AOR = \angle AOS = \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

Angle between OR and the real axis

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

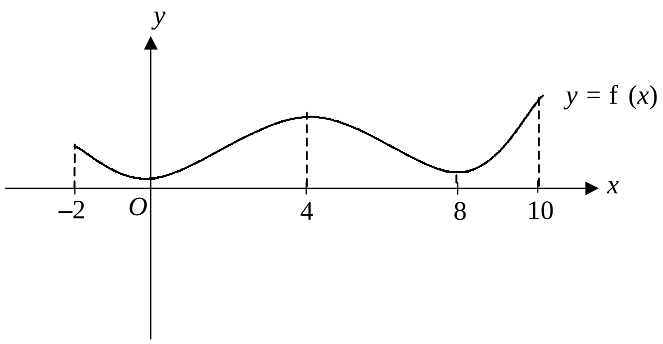
$\therefore$ the complex number represented by R is

$$\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}) .$$

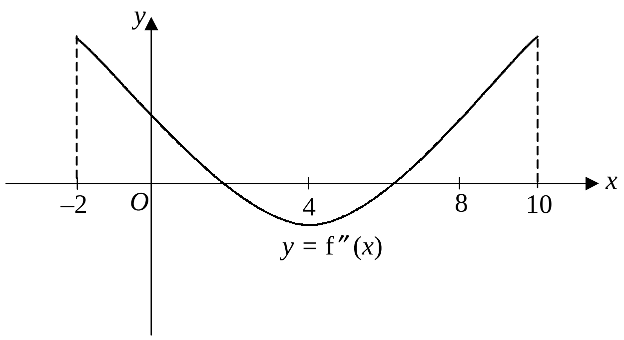
Q.18 (a) (i) $f(x)$ is increasing when $0 \le x \le 4$ and $8 \le x \le 10$.

(ii) From Figure 5 (a), $f'(x) = 0$ at $x = 0$, 4 and 8.
As $f'(x)$ changes from –ve to +ve as x increases through 0 and 8, $f(x)$ attains a minimum at $x = 0$ and 8.
As $f'(x)$ changes from +ve to –ve as x increases through 4, $f(x)$ attains a maximum at $x = 4$.

(iii)

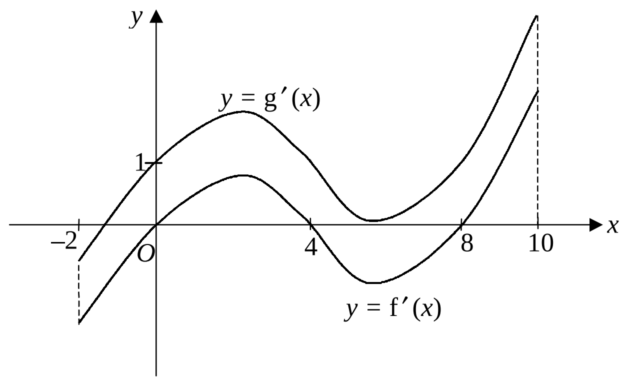


(b)



(c) (i) $g(x) = f(x) + x.$

$g'(x) = f'(x) + 1.$ - - - - - (1)



(ii) Differentiate (1) with respect to x :

$g''(x) = f''(x).$

Since $f''(x) = g''(x)$, the graphs of $y = f''(x)$ and $y = g''(x)$ are identical. The student is incorrect.