2000-CE
A MATH

PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2000

ADDITIONAL MATHEMATICS PAPER 1

8.30 am – 10.30 am (2 hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.
2. All working must be clearly shown.
3. Unless otherwise specified, numerical answers must be **exact**.
4. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.
5. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Solve $\frac{1}{x} > 1$.

(3 marks)

2. Find (a) $\frac{d}{dx}\sin^2 x$,

(b) $\frac{d}{dx}\sin^2(3x+1)$.

(4 marks)

3. (a) Show that $\frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}} = \frac{-\Delta x}{\sqrt{x}\,(\sqrt{x+\Delta x})\,(\sqrt{x}+\sqrt{x+\Delta x})}$.

(b) Find $\frac{d}{dx}(\frac{1}{\sqrt{x}})$ from first principles.

(5 marks)

4. $P(-1, 2)$ is a point on the curve $(x+2)(y+3)=5$. Find

(a) the value of $\frac{dy}{dx}$ at P,

(b) the equation of the tangent to the curve at P.

(5 marks)

5. (a) Solve $|1-x|=2$.

(b) By considering the cases $x \le 1$ and $x > 1$, or otherwise, solve $|1-x| = x-1$.

(5 marks)

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6. Express the complex number $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ in polar form.

Hence find the argument θ of $\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^{2000}$, where θ is limited to the principal values $-\pi < \theta \leq \pi$.

(6 marks)

7. α and β are the roots of the quadratic equation

$$x^2 + (p-2)x + p = 0 ,$$

where p is real.

(a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of p.

(b) If α and β are real such that $\alpha^2 + \beta^2 = 11$, find the value(s) of p.

(7 marks)

8.

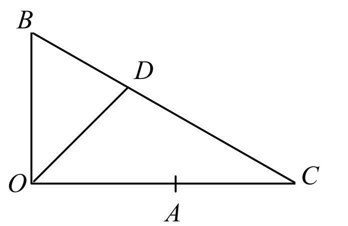

Figure 1

In Figure 1, $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$. C is a point on OA produced such that $AC = k$, where $k > 0$. D is a point on BC such that $BD : DC = 1 : 2$.

(a) Show that $\overrightarrow{OD} = \frac{1+k}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$.

(b) If $\overrightarrow{OD}$ is a unit vector, find

(i) k,

(ii) $\angle BOD$, giving your answer correct to the nearest degree.

(7 marks)

Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9.

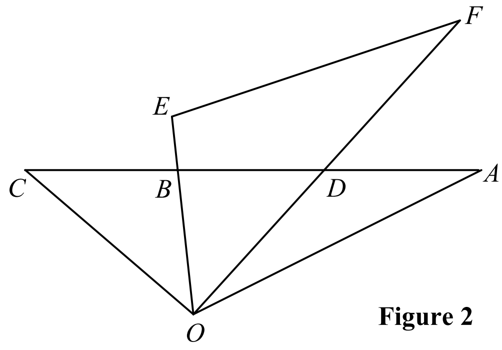

Figure 2

In Figure 2, OAC is a triangle. B and D are points on AC such that $AD = DB = BC$. F is a point on OD produced such that $OD = DF$. E is a point on OB produced such that $OE = k(OB)$, where $k > 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) Express $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Show that $\overrightarrow{OC} = -\frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$.

(iii) Express $\overrightarrow{EF}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

(5 marks)

(b) It is given that $OA = 3$, $OB = 2$ and $\angle AOB = 60°$.

(i) Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$.

(ii) Suppose that $\angle OEF = 90°$.

(1) Find the value of k.

(2) A student states that points C, E and F are collinear. Explain whether the student is correct.

(11 marks)

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10. Let $f(x)=\dfrac{7-4x}{x^2+2}$.

(a) (i) Find the x- and y-intercepts of the curve $y=f(x)$.

(ii) Find the range of values of x for which $f(x)$ is decreasing.

(iii) Show that the maximum and minimum values of $f(x)$ are 4 and $-\dfrac{1}{2}$ respectively.

(9 marks)

(b) In Figure 3, sketch the curve $y=f(x)$ for $-2\le x\le 5$.

(3 marks)

(c) Let $p=\dfrac{7-4\sin\theta}{\sin^2\theta+2}$, where θ is real.

From the graph in (b), a student concludes that the greatest and least values of p are 4 and $-\dfrac{1}{2}$ respectively. Explain whether the student is correct. If not, what should be the greatest and least values of p ?

(4 marks)

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

If you attempt Question 10, fill in the first three boxes above and tie this sheet into your answer book.

10. (b) **(continued)**

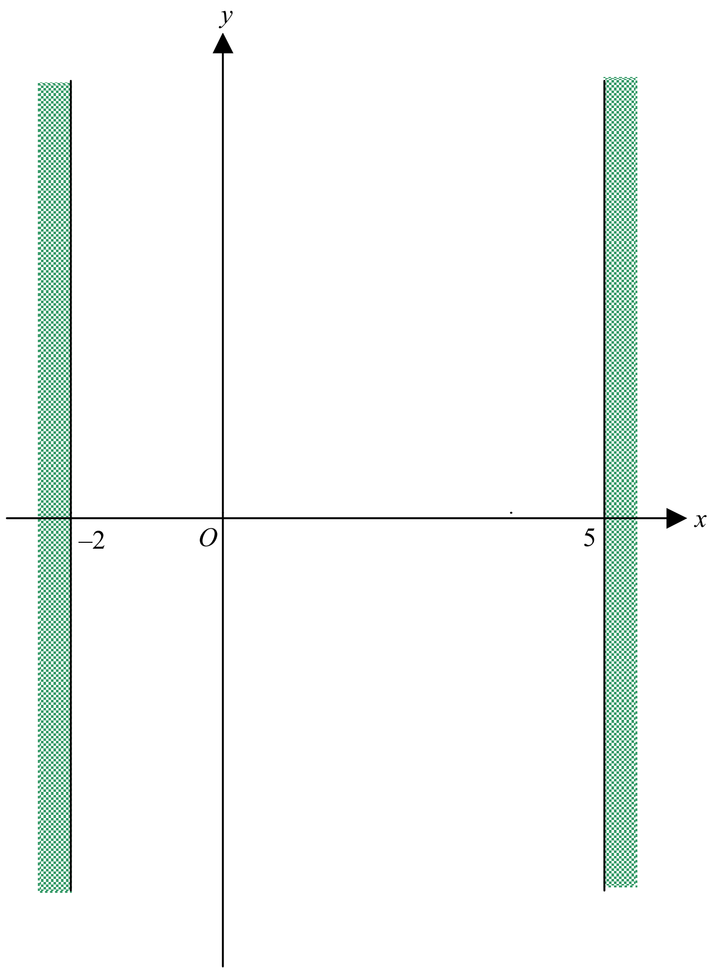

Figure 3

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11. (a) Let $w = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$. It is given that the complex number $w^2 + \frac{5}{w} - 2$ is purely imaginary.

Show that $2\cos^2\theta + 5\cos\theta - 3 = 0$.

Hence, or otherwise, find w.

(8 marks)

(b) A and B are two points in an Argand diagram representing two distinct non-zero complex numbers z_1 and z_2 respectively. Suppose that $z_2 = wz_1$, where w is the complex number found in (a).

(i) Find $\left|\frac{z_2}{z_1}\right|$ and $\arg(\frac{z_2}{z_1})$.

(ii) Let O be the point representing the complex number 0. What type of triangle is ΔOAB ? Explain your answer.

(8 marks)

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12. Consider the function $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$, where $m > \frac{1}{3}$.

(a) Show that the equation $f(x) = 0$ has distinct real roots.

(3 marks)

(b) Let α, β be the roots of the equation $f(x) = 0$, where $\alpha < \beta$.

(i) Express α and β in terms of m.

(ii) Furthermore, it is known that $4 < \beta < 5$.

(1) Show that $1 < m < \frac{6}{5}$.

(2) Figure 4 shows three sketches of the graph of $y = f(x)$ drawn by three students. Their teacher points out that the three sketches are all incorrect. Explain why each of the sketches is incorrect.

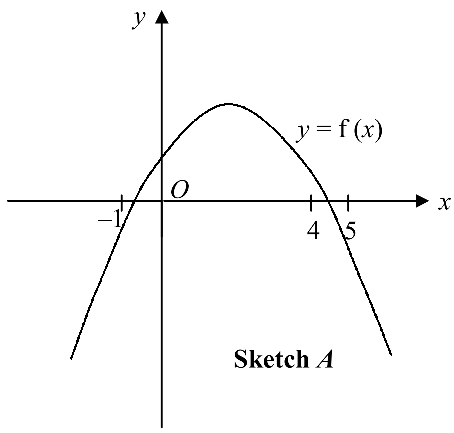
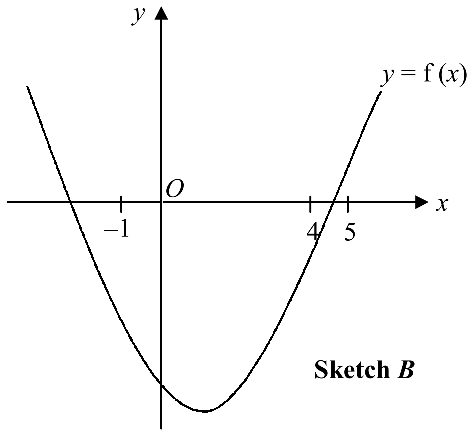


Figure 4

12. (b) (ii) (2) **(continued)**



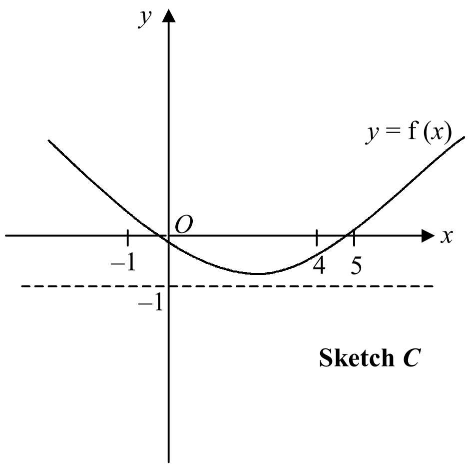


Figure 4 (continued)

(13 marks)

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13.

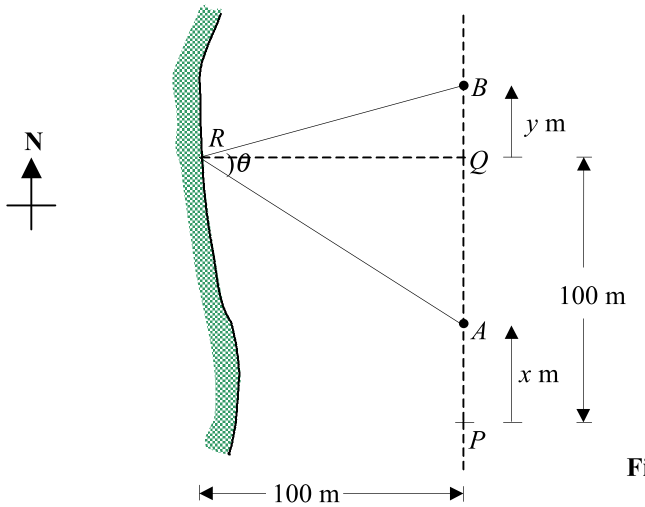


Figure 5

Two boats A and B are initially located at points P and Q in a lake respectively, where Q is at a distance 100 m due north of P. R is a point on the lakeside which is at a distance 100 m due west of Q. (See Figure 5.) Starting from time (in seconds) $t = 0$, boats A and B sail northwards. At time t, let the distances travelled by A and B be x m and y m respectively, where $0 \le x \le 100$. Let $\angle ARB = \theta$.

(a) Express $\tan \angle ARQ$ in terms of x.

Hence show that $\tan \theta = \dfrac{100\,(100 - x + y)}{10000 - 100y + xy}$. (4 marks)

(b) Suppose boat A sails with a constant speed of 2 m s^{-1} and B adjusts its speed continuously so as to keep the value of $\angle ARB$ unchanged.

(i) Using (a), show that $y = \dfrac{100x}{200 - x}$.

(ii) Find the speed of boat B at $t = 40$.

(iii) Suppose the maximum speed of boat B is 3 m s^{-1}. Explain whether it is possible to keep the value of $\angle ARB$ unchanged before boat A reaches Q.

(12 marks)

END OF PAPER

2000

Additional Mathematics

Paper 1

Section A

1. $0 < x < 1$

2. (a) $2\sin x\cos x$

   (b) $6\sin(3x+1)\cos(3x+1)$

3. (b) $\dfrac{-\sqrt{x}}{2x^2}$

4. (a) –5

   (b) $5x + y + 3 = 0$

5. (a) $x = -1$ or 3

   (b) $x \geq 1$

6. $\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}, -\dfrac{2\pi}{3}$

7. (a) $2 - p$, p

   (b) –1

8. (b) (i) $\sqrt{5} - 1$

   (ii) 48°

Section B

Q.9 (a) (i) $\overrightarrow{OD} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$

(ii) $\overrightarrow{OB} = \dfrac{\overrightarrow{OA}+2\overrightarrow{OC}}{1+2}$

$\mathbf{b} = \dfrac{\mathbf{a}+2\overrightarrow{OC}}{3}$

$\overrightarrow{OC} = -\dfrac{1}{2}\mathbf{a}+\dfrac{3}{2}\mathbf{b}$

(iii) $\overrightarrow{EF} = \overrightarrow{OF}-\overrightarrow{OE}$

$= 2\overrightarrow{OD}-k\overrightarrow{OB}$

$= 2(\dfrac{\mathbf{a}+\mathbf{b}}{2})-k\mathbf{b}$

$= \mathbf{a}+(1-k)\,\mathbf{b}$

(b) (i) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\angle AOB$

$= 3(2)\cos 60°$

$= 3$

$\mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2 = 4$

(ii) (1) $\overrightarrow{OE}\cdot\overrightarrow{EF} = 0$

$k\mathbf{b}\cdot[\,\mathbf{a}+(1-k)\,\mathbf{b}\,] = 0$

$k\mathbf{a}\cdot\mathbf{b}+k(1-k)\,\mathbf{b}\cdot\mathbf{b} = 0$

$3k+4k(1-k) = 0$

$7k-4k^2 = 0$

$k = 0$ (rejected) or $k = \dfrac{7}{4}$

$\therefore\ k = \dfrac{7}{4}$.

(2) Put $k = \frac{7}{4}$:

$\overrightarrow{EF} = \mathbf{a} + (1 - \frac{7}{4})\mathbf{b} = \mathbf{a} - \frac{3}{4}\mathbf{b}$

$\overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC}$

$= \frac{7}{4}\mathbf{b} - (-\frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b})$

$= \frac{1}{2}\mathbf{a} + \frac{1}{4}\mathbf{b}$

Since $\overrightarrow{CE} \neq \mu\, \overrightarrow{EF}$, C, E, F are not collinear. The student is incorrect.

Q.10 (a) (i) Put $x=0$, $y=\frac{7}{2}$ $\therefore$ the y-intercept is $\frac{7}{2}$.

Put $y=0$, $x=\frac{7}{4}$ $\therefore$ the x-intercept is $\frac{7}{4}$.

(ii) $\mathrm{f}(x)$ is decreasing when $\mathrm{f}'(x)\le 0$.

$$\mathrm{f}'(x)=\frac{-4(x^2+2)-(7-4x)2x}{(x^2+2)^2}$$

$$=\frac{4x^2-14x-8}{(x^2+2)^2}$$

$$\frac{4x^2-14x-8}{(x^2+2)^2}\le 0$$

$$(2x+1)(x-4)\le 0$$

$$-\frac{1}{2}\le x\le 4$$

(iii) $\mathrm{f}(x)$ is increasing when $\mathrm{f}'(x)\ge 0$,

i.e. $x\ge 4$ or $x\le -\frac{1}{2}$.

$\mathrm{f}'(x)=0$ when $x=4$ or $-\frac{1}{2}$.

As $\mathrm{f}'(x)$ changes from positive to negative as x increases through $-\frac{1}{2}$, so $\mathrm{f}(x)$ attains a maximum at $x=-\frac{1}{2}$.

At $x=-\frac{1}{2}$, $y=4$

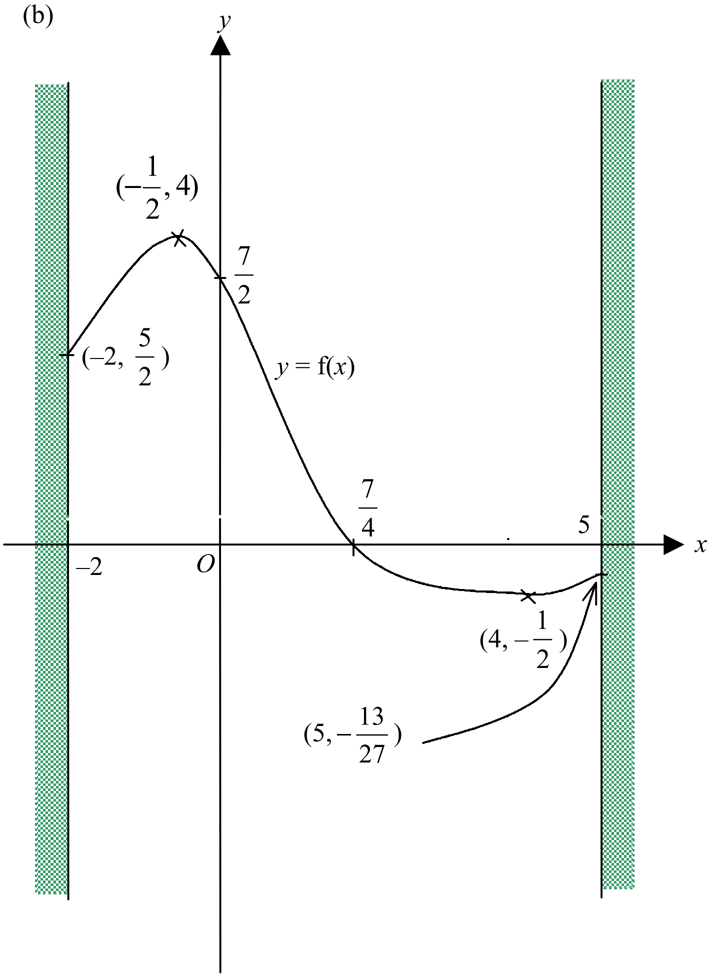
$\therefore$ the maximum value of $\mathrm{f}(x)$ is 4.

As $\mathrm{f}'(x)$ changes from negative to positive as x increases through 4, so $\mathrm{f}(x)$ attains a minimum at $x=4$.

At $x=4$, $y=-\frac{1}{2}$

$\therefore$ the minimum value of $\mathrm{f}(x)$ is $-\frac{1}{2}$.

(b)



(c) Put $x = \sin\theta$, $f(\sin\theta) = \dfrac{7 - 4\sin\theta}{\sin^2\theta + 2} = p$.

The range of possible value of $\sin\theta$ is $-1 \le \sin\theta \le 1$.
From the graph in (b), the greatest value of $f(x)$ in the range $-1 \le x \le 1$ is 4.

$\therefore$ the greatest value of p is 4 and the student is correct.

From the graph in (b), f(x) attains its least value at one of the end-points.

$f(1) = 1$, $f(-1) = \dfrac{11}{3}$.

$\therefore$ the least value of p is 1 and the student is incorrect.

Q.11 (a) $w = \cos\theta + i\sin\theta$

$w^2 = \cos 2\theta + i\sin 2\theta$

$$\frac{1}{w} = \frac{1}{\cos\theta + i\sin\theta}$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta$$

$$w^2 + \frac{5}{w} - 2$$
$$= \cos 2\theta + i\sin 2\theta + 5(\cos\theta - i\sin\theta) - 2$$
$$= \cos 2\theta + 5\cos\theta - 2 + i(\sin 2\theta - 5\sin\theta)$$

Since $w^2 + \frac{5}{w} - 2$ is purely imaginary,

$$\cos 2\theta + 5\cos\theta - 2 = 0$$
$$(2\cos^2\theta - 1) + 5\cos\theta - 2 = 0$$

$$2\cos^2\theta + 5\cos\theta - 3 = 0$$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -3$ (rejected)

$\theta = \frac{\pi}{3}$ ($\because$ $0 < \theta < \pi$)

Imaginary part

$$= \sin\frac{2\pi}{3} - 5\sin\frac{\pi}{3} \neq 0$$

$$\therefore w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

(b) (i) $\left|\frac{z_2}{z_1}\right| = |w|$

$$= 1$$

$$\arg\left(\frac{z_2}{z_1}\right) = \arg(w)$$
$$= \frac{\pi}{3}$$

(ii) $\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} = 1$

$\therefore |z_2| = |z_1|$

i.e. $OA = OB$.

$\angle AOB = \arg(z_2) - \arg(z_1)$

$= \arg(\frac{z_2}{z_1})$

$= \frac{\pi}{3}$

Since $OA = OB$, ΔOAB is isosceles.

$\angle OAB = \angle OBA = \frac{1}{2}(\pi - \frac{\pi}{3}) = \frac{\pi}{3}$

$\therefore$ ΔOAB is equilateral.

Q.12 (a) $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$

Discriminant $\Delta = (-4m)^2 + 4(5m^2 - 6m + 1)$

$= 36m^2 - 24m + 4$

$= 4(9m^2 - 6m + 1)$

$= 4(3m-1)^2 > 0 \quad (\because\ m > \frac{1}{3})$

$\therefore$ the equation $f(x) = 0$ has distinct real roots.

(b) (i) $x = \dfrac{4m \pm \sqrt{\Delta}}{2}$

$= 2m \pm (3m - 1)$

Since $\alpha < \beta$,

$\alpha = 2m - (3m-1) = -m + 1$

$\beta = 2m + (3m-1) = 5m - 1$

(ii) (1) Since $4 < \beta < 5$,

$4 < 5m - 1 < 5$

$5 < 5m < 6$

$1 < m < \frac{6}{5}$

(2) <u>Sketch *A*</u> :

Since the coefficient of x^2 in $f(x)$ is positive, the graph of $y = f(x)$ should open upwards. However, the graph in sketch *A* opens downwards, so sketch *A* is incorrect.

<u>Sketch *B*</u> :

Since $\alpha = 1 - m$ and $1 < m < \frac{6}{5}$,

$1 - 1 > 1 - m > 1 - \frac{6}{5}$

$0 > \alpha > -\frac{1}{5}$

In sketch *B*, α is less than -1, so sketch *B* is incorrect.

<u>Sketch C</u> :

$$\begin{cases} y = x^2 - 4mx - (5m^2 - 6m + 1) \\ y = -1 \end{cases}$$

$$-1 = x^2 - 4mx - (5m^2 - 6m + 1)$$

$$x^2 - 4mx - (5m^2 - 6m) = 0 \quad ---- (*)$$

Discriminant $\Delta = (-4m)^2 + 4(5m^2 - 6m)$

$$= 36m^2 - 24m$$

$$= 12m(3m - 2)$$

Since $1 < m < \frac{6}{5}$, $\Delta > 0$.

As $\Delta > 0$, equation (*) has real roots,
i.e. $y = \mathrm{f}(x)$ and $y = -1$ always have intersecting points. However, the line and the graph in sketch C do not intersect, so sketch C is incorrect.

Q.13 (a) $\tan \angle ARQ = \frac{100-x}{100}$

$\tan\theta = \tan(\angle ARQ + \angle QRB)$

$= \frac{\tan \angle ARQ + \tan \angle QRB}{1-(\tan \angle ARQ)(\tan \angle QRB)}$

$= \frac{\frac{100-x}{100} + \frac{y}{100}}{1-(\frac{100-x}{100})(\frac{y}{100})}$

$= \frac{100(100-x+y)}{10000-100y+xy}$

(b) (i) At $t=0$, $\tan\theta = \frac{PQ}{RQ}$

$= \frac{100}{100} = 1.$

Since $\angle ARB$ remains unchanged,

$\frac{100(100-x+y)}{10000-100y+xy} = 1$

$10000-100x+100y = 10000-100y+xy$

$200y - xy = 100x$

$y = \frac{100x}{200-x}$

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(200-x)(100)-100x(-1)}{(200-x)^2}\frac{\mathrm{d}x}{\mathrm{d}t}$

$= \frac{20000}{(200-x)^2}\frac{\mathrm{d}x}{\mathrm{d}t}$

$= \frac{40000}{(200-x)^2}$

At $t=40$, $x = 40\times 2 = 80$.

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{40000}{(200-80)^2}$

$= \frac{25}{9}$

$\therefore$ the speed of boat B at $t=40$ is $\frac{25}{9}$ m s^{-1}.

(iii) From (ii), $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{40000}{(200-x)^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}t}\le 3$$

$$\frac{40000}{(200-x)^2}\le 3$$

$$200-x\ge\frac{200}{\sqrt{3}}$$

$$x\le 200(1-\frac{1}{\sqrt{3}})$$

When $x>200(1-\frac{1}{\sqrt{3}})$, $\frac{\mathrm{d}y}{\mathrm{d}t}>3$.

So it is impossible to keep $\angle ARB$ unchanged before boat A reaches Q.

2000-CE
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PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2000

ADDITIONAL MATHEMATICS PAPER 2

11.15 am – 1.15 pm (2 hours)
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$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Find $\int \sqrt{2x+1}\,\mathrm{d}x$.

(4 marks)

2. Expand $(1+2x)^7(2-x)^2$ in ascending powers of x up to the term x^2.

(5 marks)

3.

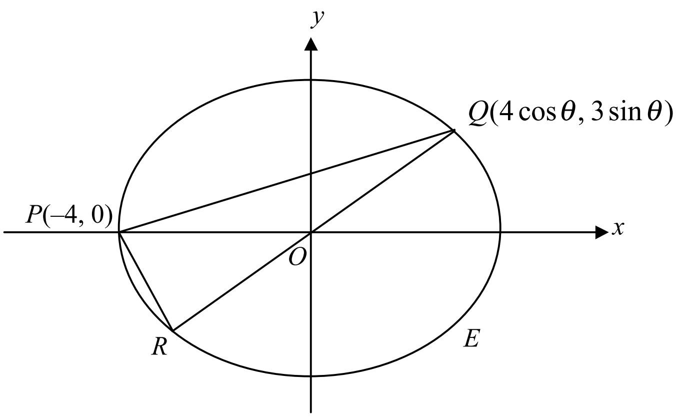

Figure 1

Figure 1 shows the ellipse $E : \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. $P(-4, 0)$ and $Q(4\cos\theta, 3\sin\theta)$ are points on E, where $0 < \theta < \dfrac{\pi}{2}$. R is a point such that the mid-point of QR is the origin O.

(a) Write down the coordinates of R in terms of θ.

(b) If the area of ΔPQR is 6 square units, find the coordinates of Q.

(6 marks)

4. Prove, by mathematical induction, that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

for all positive integers n.

(6 marks)

5.

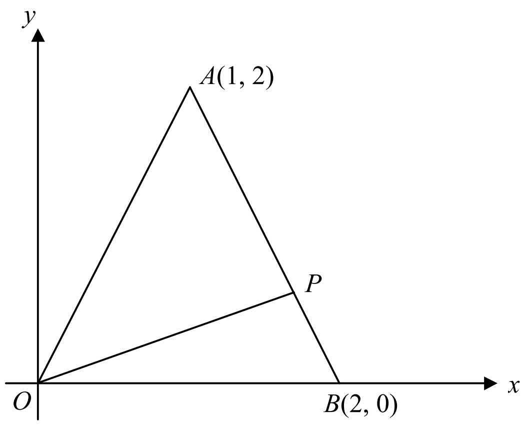

Figure 2

In Figure 2, the coordinates of points A and B are $(1, 2)$ and $(2, 0)$ respectively. Point P divides AB internally in the ratio $1 : r$.

(a) Find the coordinates of P in terms of r.

(b) Show that the slope of OP is $\frac{2r}{2+r}$.

(c) If $\angle AOP = 45°$, find the value of r.

(6 marks)

6.

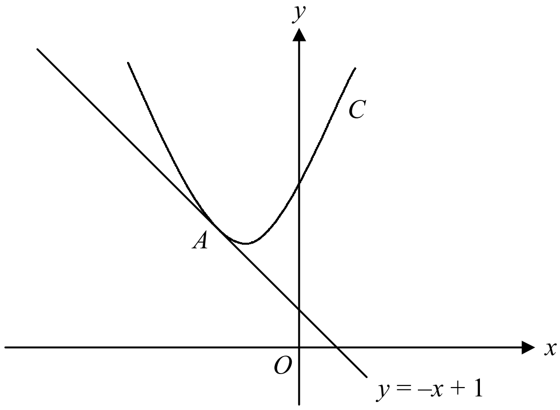

Figure 3

The slope at any point (x, y) of a curve C is given by $\frac{dy}{dx} = 2x + 3$. The line $y = -x + 1$ is a tangent to the curve at point A. (See Figure 3.) Find

(a) the coordinates of A,

(b) the equation of C.

(7 marks)

7. (a) By expressing $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x + \theta)$, or otherwise, find the general solution of the equation

$$\cos x - \sqrt{3}\sin x = 2 .$$

(b) Find the number of points of intersection of the curves $y = \cos x$ and $y = 2 + \sqrt{3}\sin x$ for $0 < x < 9\pi$.

(8 marks)

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Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. (a) Find $\int \cos 3x \cos x \, dx$.

(3 marks)

(b) Show that $\dfrac{\sin 5x - \sin x}{\sin x} = 4\cos 3x \cos x$.

Hence, or otherwise, find $\int \dfrac{\sin 5x}{\sin x} \, dx$.

(4 marks)

(c) Using a suitable substitution, show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x} \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x} \, dx .$$

(4 marks)

(d)

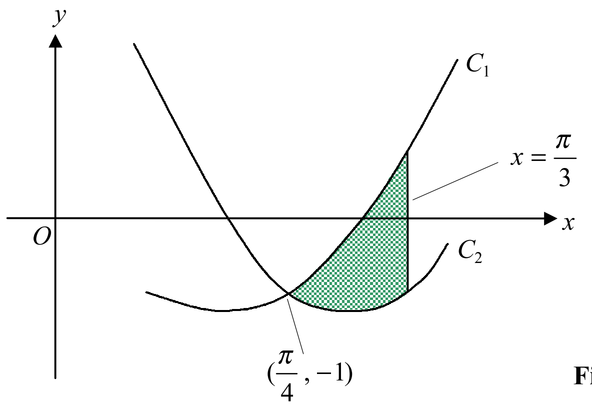

Figure 4

In Figure 4, the curves $C_1 : y = \dfrac{\cos 5x}{\cos x}$ and $C_2 : y = \dfrac{\sin 5x}{\sin x}$ intersect at the point $(\dfrac{\pi}{4}, -1)$. Find the area of the shaded region bounded by C_1, C_2 and the line $x = \dfrac{\pi}{3}$.

(5 marks)

9. Given a family of circles

$$F \ : \ x^2 + y^2 + (4k+4)x + (3k+1)y - (8k+8) = 0 \, ,$$

where k is real. C_1 is the circle $x^2 + y^2 - 2y = 0$.

(a) Show that

(i) C_1 is a circle in F,

(ii) C_1 touches the x-axis.

(4 marks)

(b) Besides C_1, there is another circle C_2 in F which also touches the x-axis.

(i) Find the equation of C_2.

(ii) Show that C_1 and C_2 touch externally.

(7 marks)

(c)

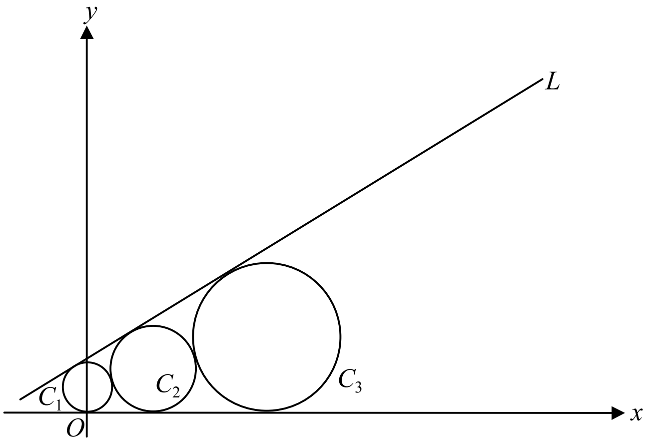

Figure 5

Figure 5 shows the circles C_1 and C_2 in (b). L is a common tangent to C_1 and C_2. C_3 is a circle touching C_2, L and the x-axis but it is not in F. (See Figure 5.) Find the equation of C_3.

(Hint : The centres of the three circles are collinear.)

(5 marks)

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10.

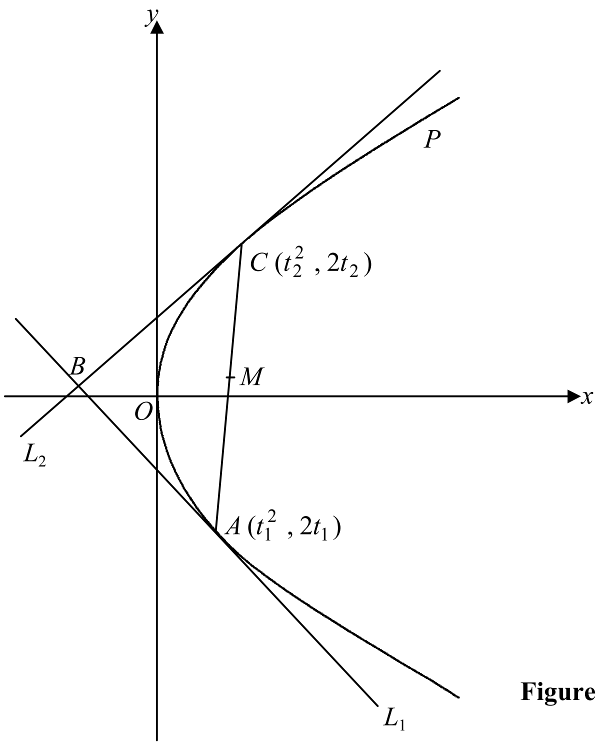

Figure 6(a)

Figure 6(a) shows a parabola $P: y^2 = 4x$. $A(t_1^2, 2t_1)$ and $C(t_2^2, 2t_2)$ are two distinct points on P, where $t_1 < 0 < t_2$. L_1 and L_2 are tangents to P at A and C respectively and they intersect at point B. Let M be the mid-point of AC.

(a) Show that

(i) the equation of L_1 is $x - t_1 y + t_1^2 = 0$,

(ii) the coordinates of B are $(t_1 t_2, t_1 + t_2)$,

(iii) BM is parallel to the x-axis.

(7 marks)

(b)

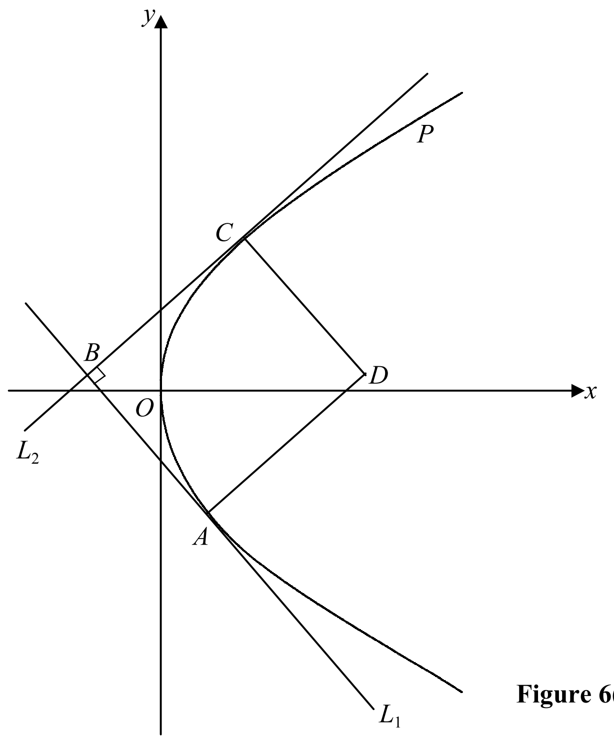

Figure 6(b)

Suppose L_1 and L_2 are perpendicular to each other and D is a point such that $ABCD$ is a rectangle. (See Figure 6(b).)

(i) Find the value of $t_1 t_2$.

(ii) Show that the coordinates of D are $(t_1^2 + t_2^2 + 1,\ t_1 + t_2)$.

(iii) Find the equation of the locus of D as A and C move along the parabola P.

(9 marks)

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11. (a)

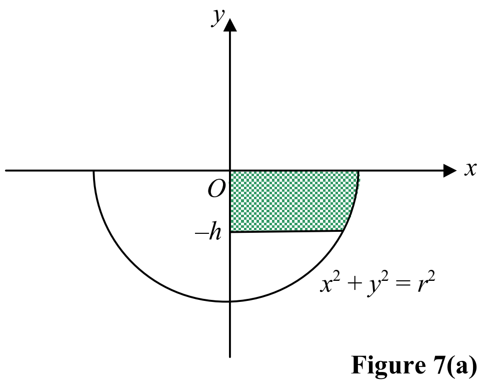


Figure 7(a)

In Figure 7 (a), the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the x-axis, the y-axis and the line $y = -h$, where $h > 0$. If the shaded region is revolved about the y-axis, show that the volume of the solid generated is $(r^2 h - \frac{1}{3} h^3)\pi$ cubic units.

(4 marks)

(b)

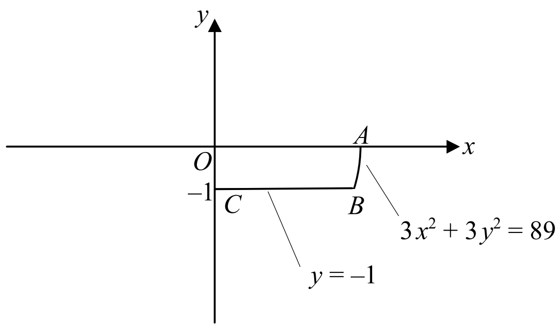


Figure 7(b)

In Figure 7 (b), A and C are points on the x-axis and y-axis respectively, AB is an arc of the circle $3x^2 + 3y^2 = 89$ and BC is a segment of the line $y = -1$. A mould is formed by revolving AB and BC about the y-axis. Using (a), or otherwise, show that the capacity of the mould is $\frac{88\pi}{3}$ cubic units.

(2 marks)

(c)

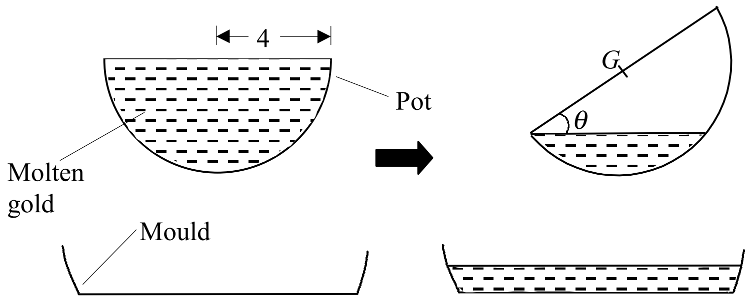


Figure 7(c) **Figure 7(d)**

A hemispherical pot of inner radius 4 units is completely filled with molten gold. (See Figure 7 (c).) The molten gold is then poured into the mould mentioned in (b) by steadily tilting the pot. Suppose the pot is tilted through an angle θ and G is the centre of the rim of the pot. (See Figure 7 (d).)

(i) Find, in terms of θ ,

(1) the distance between G and the surface of the molten gold remaining in the pot,

(2) the volume of gold poured into the mould.

(ii) When the mould is completely filled with molten gold, show that

$$8\sin^3\theta - 24\sin\theta + 11 = 0 .$$

Hence find the value of θ .

(10 marks)

12. (a)

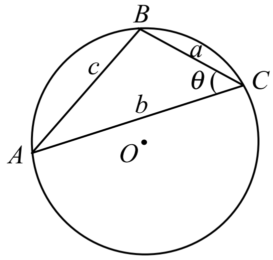


Figure 8(a)

In Figure 8 (a), a triangle ABC is inscribed in a circle with centre O and radius r. $AB = c$, $BC = a$ and $CA = b$. Let $\angle BCA = \theta$.

(i) Express $\cos\theta$ in terms of a, b and c.

(ii) Show that $r = \dfrac{c}{2\sin\theta}$.

(iii) Using (i) and (ii), or otherwise, show that

$$r = \frac{abc}{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}.$$

(7 marks)

(b) **In this part, numerical answers should be given correct to two significant figures.**

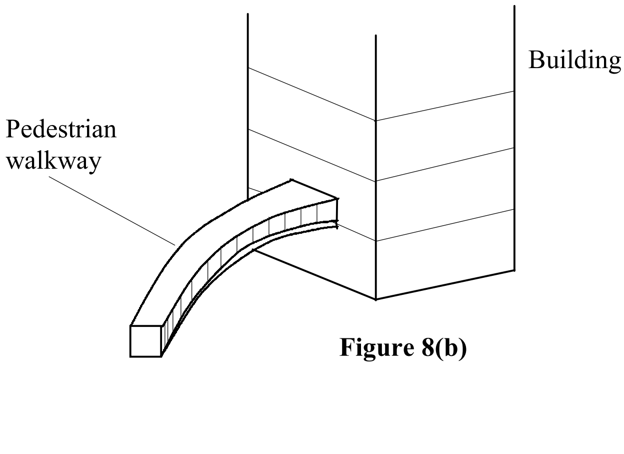


Figure 8(b)

12. (b) **(continued)**

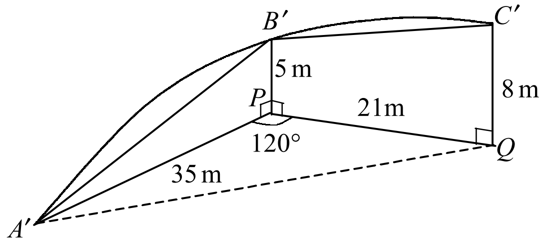


Figure 8(c)

Figure 8 (b) shows a pedestrian walkway joining the horizontal ground and the first floor of a building. To estimate its length, the walkway is modelled by a circular arc $A'B'C'$ as shown in Figure 8 (c), where A' denotes the entrance to the walkway on the ground and C' the exit leading to the first floor of the building. P and Q are the feet of perpendiculars from B' and C' to the ground respectively. It is given that $A'P = 35$ m, $PQ = 21$ m, $B'P = 5$ m, $C'Q = 8$ m and $\angle A'PQ = 120°$.

(i) Find the radius of the circular arc $A'B'C'$.

(ii) Estimate the length of the walkway.

(9 marks)

END OF PAPER

2000

Additional Mathematics

Paper 2

Section A

1. $\frac{1}{3}(2x+1)^{\frac{3}{2}}+c$, where c is a constant

2. $4+52x+281x^2+\cdots$

3. (a) $(-4\cos\theta, -3\sin\theta)$

   (b) $(2\sqrt{3}, \frac{3}{2})$

5. (a) $(\frac{2+r}{1+r}, \frac{2r}{1+r})$

   (c) $\frac{2}{5}$

6. (a) $(-2, 3)$

   (b) $y=x^2+3x+5$

7. (a) $x=2n\pi-\frac{\pi}{3}$, where n is an integer

   (b) 4

Section B

Q.8 (a) $\int \cos 3x \cos x \, dx = \int \frac{1}{2}(\cos 4x + \cos 2x)\, dx$

$$= \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + c \text{, where } c \text{ is a constant}$$

(b) $\frac{\sin 5x - \sin x}{\sin x} = \frac{2\sin 2x \cos 3x}{\sin x}$

$$= \frac{2(2\sin x \cos x)\cos 3x}{\sin x}$$

$$= 4\cos x \cos 3x$$

$$\int \frac{\sin 5x}{\sin x}\, dx = \int (1 + 4\cos 3x \cos x)\, dx$$

$$= x + 4\left(\frac{\sin 4x}{8} + \frac{\sin 2x}{4}\right) + c \text{, where } c \text{ is a constant}$$

$$= x + \frac{1}{2}\sin 4x + \sin 2x + c$$

(c) Put $x = \frac{\pi}{2} - \theta$:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x}\, dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin 5(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)}(-d\theta)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5\theta}{\cos\theta}\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x}\, dx$$

(d) Area of shaded region

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\cos 5x}{\cos x} - \frac{\sin 5x}{\sin x}\right) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x}\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x}\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\, dx \quad \text{(using (c))}$$

$$= \left[x + \frac{1}{2}\sin 4x + \sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \left[x + \frac{1}{2}\sin 4x + \sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 2 - \sqrt{3}$$

Q.9 (a) (i) Put $k = -1$ into F, the equation becomes

$$x^2 + y^2 + (-4+4)x + (-3+1)y - (-8+8) = 0$$

i.e. $x^2 + y^2 - 2y = 0$.

$\therefore$ C_1 is a circle in F.

(ii) Co-ordinates of centre of $C_1 = (0, 1)$.

radius of $C_1 = 1$

Since the y-coordinate of centre is equal to the radius, C_1 touches the x-axis.

(b) (i) Put $y = 0$ in F :

$$x^2 + (4k+4)x - (8k+8) = 0$$

Since the circle touches the x-axis,

$$(4k+4)^2 + 4(8k+8) = 0$$

$$16k^2 + 64k + 48 = 0$$

$$16(k+1)(k+3) = 0$$

$k = -1$ (rejected) or $k = -3$,

$\therefore$ the equation of C_2 is

$$x^2 + y^2 + [4(-3)+4]x + [3(-3)+1]y - [(-3)\times 8 + 8] = 0$$

$$x^2 + y^2 - 8x - 8y + 16 = 0$$

(ii) Co-ordinates of centre of $C_1 = (0, 1)$, radius $= 1$.

Co-ordinates of centre of $C_2 = (4, 4)$, radius $= 4$.

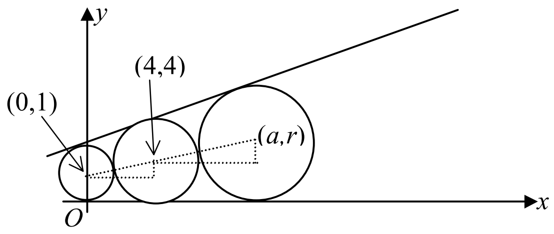
Distance between centres $= \sqrt{(4-0)^2 + (4-1)^2}$

$= 5$

$=$ sum of radii of C_1 and C_2

$\therefore$ C_1 and C_2 touch externally.

(c) Let radius of C_3 be r and coordinates of its centre be (a, r).



Considering the similar triangles,

$$\frac{r+4}{1+4}=\frac{r-4}{4-1}$$

$$3r+12=5r-20$$

$$r=16$$

$$\frac{a-4}{4-0}=\frac{r+4}{4+1}$$

$$=\frac{16+4}{4+1}$$

$$a=20$$

$\therefore$ the equation of C_3 is $(x-20)^2+(y-16)^2=256$.

Q.10 (a) (i) $y^2 = 4x$

$2y\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{2}{y}$

At point A, $\frac{dy}{dx} = \frac{2}{2t_1} = \frac{1}{t_1}$

Equation of L_1 is

$\frac{y-2t_1}{x-t_1^2} = \frac{1}{t_1}$

$t_1 y - 2t_1^2 = x - t_1^2$

$x - t_1 y + t_1^2 = 0$

(ii) Equation of L_2 is $x - t_2 y + t_2^2 = 0$.

$$\begin{cases} x - t_1 y + t_1^2 = 0 & ------(1) \\ x - t_2 y + t_2^2 = 0 & ------(2) \end{cases}$$

(1) – (2) : $(t_2 - t_1)y + (t_1^2 - t_2^2) = 0$

$y = t_1 + t_2$

$x = t_1(t_1 + t_2) - t_1^2 = t_1 t_2$

$\therefore$ the coordinates of B are $(t_1 t_2, t_1 + t_2)$.

(iii)The coordinates of M are $(\frac{t_1^2 + t_2^2}{2}, \frac{2t_1 + 2t_2}{2})$,

i.e. $(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2)$.

As the y-coordinates of B and M are equal,
BM is parallel to the x-axis.

(b) (i) (Slope of L_1) (Slope of L_2) = –1

$(\frac{1}{t_1})(\frac{1}{t_2}) = -1$

$t_1 t_2 = -1$

(ii) Since $ABCD$ is a rectangle, mid-point of BD coincides with mid-point of AC, i.e. point M.

$$\frac{x+t_1t_2}{2}=\frac{t_1^{\ 2}+t_2^{\ 2}}{2}$$

$$x=t_1^{\ 2}+t_2^{\ 2}-t_1t_2$$

$$=t_1^{\ 2}+t_2^{\ 2}+1 \quad (\because\ t_1t_2=-1)$$

Since BD is parallel to the x-axis, the y-coordinate of D = y-coordinate of B = $t_1 + t_2$.

$\therefore$ the coordinates of D are $(t_1^{\ 2}+t_2^{\ 2}+1\,,\,t_1+t_2)$.

(iii) Let (x, y) be the coordinates of D.

$$\begin{cases} x=t_1^{\ 2}+t_2^{\ 2}+1 \\ y=t_1+t_2 \end{cases}$$

$$x=(t_1+t_2)^2-2t_1t_2+1$$

$$=y^2-2(-1)+1$$

$$x=y^2+3$$

$\therefore$ the equation of the locus is $x - y^2 - 3 = 0$.

Q.11 (a) Volume $= \int_{-h}^{0} \pi x^2 \, dy$

$= \int_{-h}^{0} \pi (r^2 - y^2) \, dy$

$= \pi [r^2 y - \frac{1}{3} y^3]_{-h}^{0}$

$= \pi (r^2 h - \frac{1}{3} h^3)$ cubic units

(b) Put $h = 1, r = \sqrt{\frac{89}{3}}$:

Using (a),

capacity of the mould $= \pi [\frac{89}{3}(1) - \frac{1}{3}(1)^3]$

$= \frac{88\pi}{3}$ cubic units

(c) (i) (1) Distance $= 4\sin\theta$.

(2) Put $r = 4, h = 4\sin\theta$.

Using (a), amount of gold poured into the pot

$= \pi[4^2(4\sin\theta) - \frac{1}{3}(4\sin\theta)^3]$

$= \pi(64\sin\theta - \frac{64}{3}\sin^3\theta)$

(ii) When the mould is completely filled,

$\pi(64\sin\theta - \frac{64}{3}\sin^3\theta) = \frac{88\pi}{3}$

$64\sin^3\theta - 192\sin\theta + 88 = 0$

$8\sin^3\theta - 24\sin\theta + 11 = 0$ ---- (*)

Put $\sin\theta = \frac{1}{2}$:

$8\sin^3\theta - 24\sin\theta + 11 = 0.$

$\therefore \sin\theta = \frac{1}{2}$ is a root of (*)

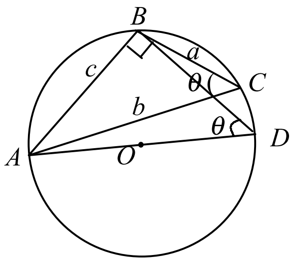
$(2\sin\theta - 1)(4\sin^2\theta + 2\sin\theta - 11) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-2 \pm \sqrt{180}}{8}$ (rejected)

$\therefore \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Q.12 (a)



(i) $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab}$

(ii) Consider ΔABD :

$AD = 2r$

$\angle BDA = \angle BCA = \theta$

$\angle ABD = 90°$

$\therefore \sin\theta = \dfrac{c}{2r}$

$r = \dfrac{c}{2\sin\theta}$

(iii) $\sin^2\theta + \cos^2\theta = 1$

$(\dfrac{c}{2r})^2 + (\dfrac{a^2 + b^2 - c^2}{2ab})^2 = 1$

$\dfrac{c^2}{4r^2} = 1 - \dfrac{(a^2 + b^2 - c^2)^2}{4a^2b^2}$

$r^2 = \dfrac{a^2b^2c^2}{4a^2b^2 - (a^2 + b^2 - c^2)^2}$

$r = \dfrac{abc}{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}$

(b) (i) Consider $\Delta A'B'C'$: $A'B' = \sqrt{(A'P)^2 + (PB')^2}$

$= \sqrt{35^2 + 5^2}$

$= \sqrt{1250}$

$B'C' = \sqrt{(PQ)^2 + (QC' - PB')^2}$

$= \sqrt{21^2 + (8-5)^2}$

$= \sqrt{450}$

$$(A'Q)^2 = (A'P)^2 + (PQ)^2 - 2(A'P)(PQ)^2 \cos\angle A'PQ$$
$$= 35^2 + 21^2 - 2(35)(21)\cos 120°$$
$$= 2401$$
$$A'C' = \sqrt{(A'Q)^2 + (QC')^2}$$
$$= \sqrt{2401 + 8^2}$$
$$= \sqrt{2465}$$

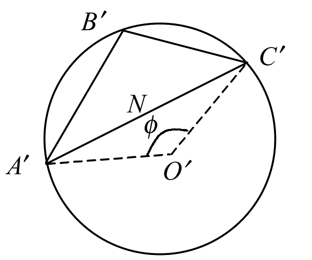
Using (a) (iii), put $a = \sqrt{450}$, $b = \sqrt{2465}$, $c = \sqrt{1250}$:

$$r = \frac{\sqrt{1250}\sqrt{450}\sqrt{2465}}{\sqrt{4(450)(2465) - (450 + 2465 - 1250)^2}}$$

= 29 m (correct to 2 sig. figures)

$\therefore$ the radius of arc $A'B'C'$ is 29 m.

(ii)



Let O' be the centre of the circle passing through A', B' and C',
ϕ be the angle subtended by arc $A'B'C'$ at O'.

Consider $\Delta O'A'N$ (N is the mid-point of $A'C'$)

$$\sin\frac{\phi}{2} = \frac{\frac{1}{2}A'C'}{r}$$

$$\sin\frac{\phi}{2} = \frac{\frac{1}{2}\sqrt{2465}}{28.86}$$
$$= 0.8602$$

$\phi = 2.07$

Length of walkway

$= \text{length of } \widehat{A'B'C'} = r\phi$
$= 28.86\ (2.07)$
$= 60$ m (correct to 2 sig. figures)

$\therefore$ the length of the walkway is 60 m.