99-CE
A MATHS
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1999

ADDITIONAL MATHEMATICS PAPER 1

8.30 am – 10.30 am (2 hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.

2. All working must be clearly shown.

3. Unless otherwise specified, numerical answers must be **exact**.

4. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.

5. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Find (a) $\frac{d}{dx}\sin(x^2+1)$,

(b) $\frac{d}{dx}[\frac{\sin(x^2+1)}{x}]$.

(4 marks)

2. Solve the inequality $\frac{x}{x-1} > 2$.

(4 marks)

3. Solve $|x-3| = |x^2-4x+3|$.

(5 marks)

4. Let $f(x) = 2x^2 + 2(k-4)x + k$, where k is real.

(a) Find the discriminant of the equation $f(x) = 0$.

(b) If the graph of $y = f(x)$ lies above the x-axis for all values of x, find the range of possible values of k.

(5 marks)

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5. Express $1+i$ in polar form.

Hence find the three cube roots of $1+i$, giving your answers in polar form.

(5 marks)

6. The point $P(a,a)$ is on the curve $3x^2 - xy - y^2 - a^2 = 0$, where a is a non-zero constant.

(a) Find the value of $\frac{dy}{dx}$ at P.

(b) Find the equation of the tangent to the curve at P.

(6 marks)

7. Let $\mathbf{a}$, $\mathbf{b}$ be two vectors such that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{b}| = 4$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.

(a) Find $|\mathbf{a}|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If the vector $(m\mathbf{a} + \mathbf{b})$ is perpendicular to $\mathbf{b}$, find the value of m.

(6 marks)

8.

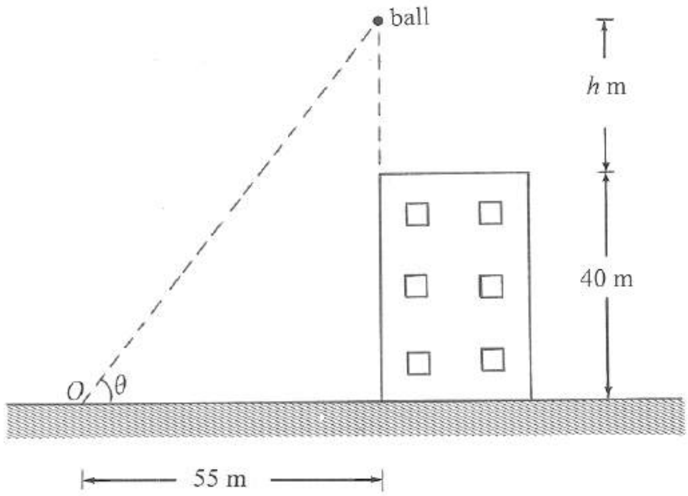

Figure 1

A ball is thrown vertically upwards from the roof of a building 40 metres in height. After t seconds, the height of the ball above the roof is h metres, where $h = 20t - 5t^2$. At this instant, the angle of elevation of the ball from a point O, which is at a horizontal distance of 55 metres from the building, is θ. (See Figure 1.)

(a) Find (i) $\tan\theta$ in terms of t,

(ii) the value of θ when $t = 3$.

(b) Find the rate of change of θ with respect to time when $t = 3$.

(7 marks)

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Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9. Let $f(x) = a\sin 2x + b\cos x$, where $0 \le x \le \pi$ and a, b are constants.
Figure 2 (a) shows the graph of $y = f'(x)$.

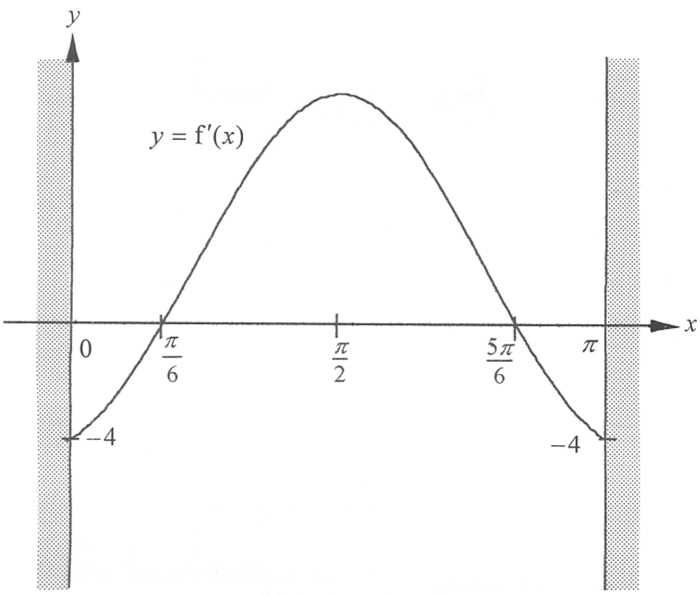

Figure 2(a)

(a) (i) Find $f'(x)$ in terms of a, b and x.

(ii) Using Figure 2(a), show that $a = -2$ and $b = -4$.
(4 marks)

(b) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the maximum and minimum points of the curve $y = f(x)$.
(7 marks)

(c) In Figure 2 (b) , sketch the curve $y = f(x)$. (3 marks)

(d) Let $g(x) = |a\sin 2x + b\cos x - 6|$, where $0 \le x \le \pi$. Using the result of (c), write down the range of possible values of $g(x)$.
(2 marks)

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

If you attempt Question 9, fill in the details in the first three boxes above and tie this sheet into your answer book.

9. (c) **(continued)**

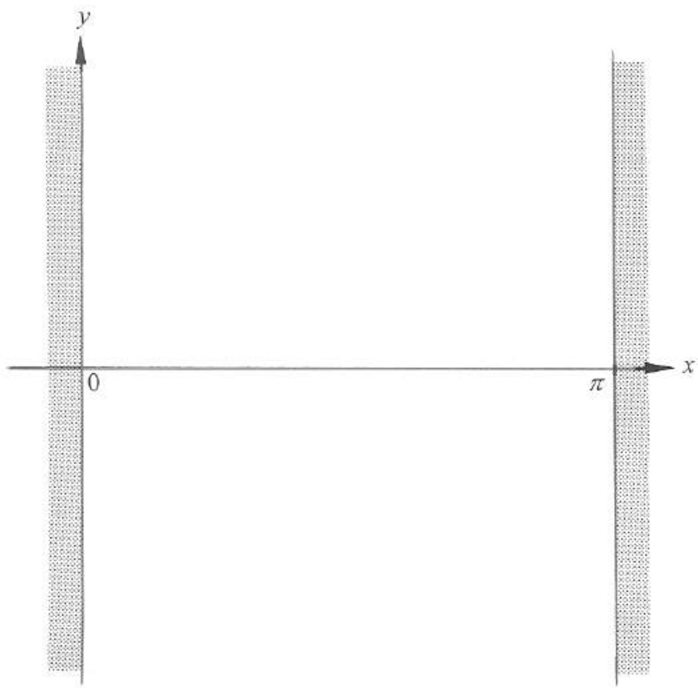

Figure 2(b)

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10.

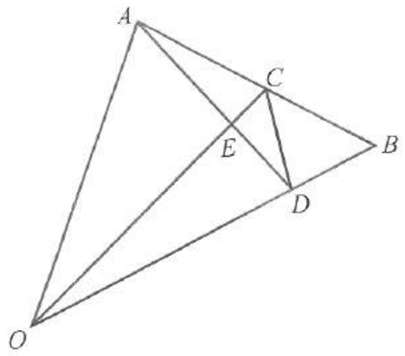

Figure 3

In Figure 3, OAB is a triangle. C and D are points on AB and OB respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. OC and AD intersect at a point E. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(4 marks)

(b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

(i) Express $\overrightarrow{OE}$ in terms of r, $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

Hence show that $r = \dfrac{6}{7}$ and $k = \dfrac{3}{5}$.

(6 marks)

(c) It is given that $EC : ED = 1 : 2$.

(i) Using (b), or otherwise, find $EA : EO$.

(ii) Explain why $OACD$ is a cyclic quadrilateral.

(6 marks)

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11. Figure 4 shows a parallelogram $OABC$ in an Argand diagram. $OA = 2$ and OA makes an angle $60°$ with the positive real axis. Let z_1 , z_2 and z_3 be the complex numbers represented by vertices A, B and C respectively. It is given that $z_3 = (\sqrt{3}i)\, z_1$.

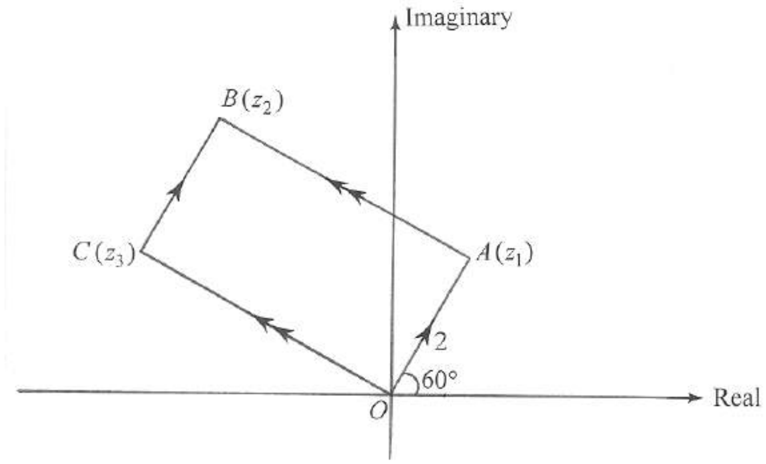


Figure 4

(a) Find z_1 and z_3 in standard form.

(3 marks)

(b) Show that $\dfrac{z_2}{z_1} = 1 + \sqrt{3}i$.

Hence, or otherwise, find $\angle AOB$.

(5 marks)

(c) Let $w = \cos\theta + i\sin\theta$, where $0° \le \theta < 360°$. Point E is a point in the Argand diagram representing the complex number wz_3 . Find the value(s) of θ in each of the following cases :

(i) E represents the complex number $\bar{z}_3$.

(ii) Points E, O and A lie on the same straight line.

(8 marks)

12.

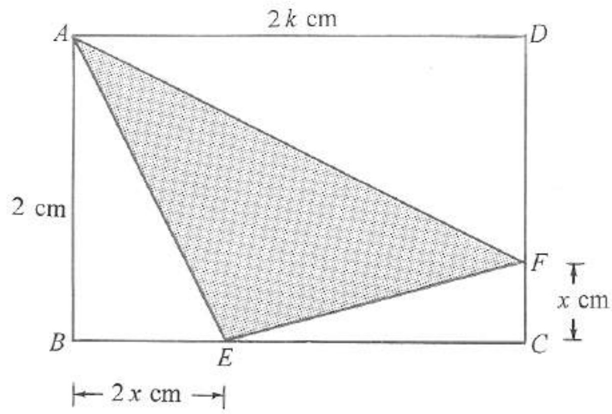


Figure 5

Figure 5 shows a rectangle $ABCD$ with $AB = 2$ cm and $AD = 2k$ cm, where k is a positive number. E and F are two variable points on the sides BC and CD respectively such that $CF = x$ cm and $BE = 2x$ cm, where x is a non-negative number. Let S cm^2 denote the area of ΔAEF.

(a) Show that $S = x^2 - 2x + 2k$. (3 marks)

(b) Suppose $k = \frac{3}{2}$.

(i) By considering that points E and F lie on the sides BC and CD respectively, show that $0 \le x \le \frac{3}{2}$.

(ii) Find the least value of S and the corresponding value of x.

(iii) Find the greatest value of S. (9 marks)

(c) Suppose $k = \frac{3}{8}$. A student says that S is least when $x = 1$.

(i) Explain whether the student is correct.

(ii) Find the least value of S. (4 marks)

13.

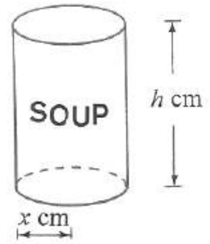


Figure 6

A food company produces cans of instant soup. Each can is in the form of a right cylinder with a base radius of x cm and a height of h cm (see Figure 6) and its capacity is V cm^3, where V is constant. The cans are made of thin metal sheets. The cost of the curved surface of the can is 1 cent per cm^2 and the cost of the plane surfaces is k cents per cm^2. Let C cents be the production cost of one can. For economic reasons, the value of C is minimised.

(a) Express h in terms of π, x and V.

Hence show that $C = \frac{2V}{x} + 2\pi k x^2$. (3 marks)

(b) If $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$, express x^3 in terms of π, k and V.

Hence show that C is a minimum when $\frac{x}{h} = \frac{1}{2k}$. (6 marks)

(c) Suppose $k = 2$ and $V = 256\pi$.

(i) Find the values of x and h.

(ii) If the value of k increases, how would the dimensions of the can be affected ? Explain your answer. (5 marks)

(d) The company intends to produce a bigger can of capacity $2V$ cm^3, which is also in the form of a right cylinder. Suppose the costs of the curved surface and plane surfaces of the bigger can are maintained at 1 cent and k cents per cm^2 respectively. A worker suggests that the ratio of base radius to height of the bigger can should be twice that of the smaller can in order to minimize the production cost. Explain whether the worker is correct. (2 marks)

END OF PAPER

Outlines of Solutions

1999 Additional Mathematics

Paper 1

Section A

1. (a) $2x\cos(x^2+1)$

   (b) $2\cos(x^2+1)-\frac{1}{x^2}\sin(x^2+1)$

2. $1< x<2$

3. 0, 2 or 3

4. (a) $4k^2-40k+64$

   (b) $2<k<8$

5. $\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

   $2^{\frac{1}{6}}\left[\cos\left(\frac{2k\pi}{3}+\frac{\pi}{12}\right)+i\sin\left(\frac{2k\pi}{3}+\frac{\pi}{12}\right)\right]$, $k=-1, 0, 1$

6. (a) $\frac{5}{3}$

   (b) $5x-3y-2a=0$

7. (a) 5

   (b) 10

   (c) –1.6

8. (a) (i) $\tan\theta=\frac{4t-t^2+8}{11}$

   (ii) $\frac{\pi}{4}$

   (b) $-\frac{1}{11}\text{s}^{-1}$

Section B

Q.9 (a) (i) $f'(x) = 2a\cos 2x - b\sin x$

(ii) From figure 2 (a), $f'(0) = -4$ and $f'(\frac{\pi}{6}) = 0$

$2a\cos 0 - b\sin 0 = -4$
$a = -2$

$2(-2)\cos\frac{\pi}{3} - b\sin\frac{\pi}{6} = 0$

$b = -4$

(b) (i) $f(0) = -4$ $\therefore$ the y-intercept is -4.

Put $f(x) = 0$: $-2\sin 2x - 4\cos x = 0$

$-4\sin x\cos x - 4\cos x = 0$

$-4\cos x(1 + \sin x) = 0$

$\cos x = 0$ or $\sin x = -1$ (rejected)

$x = \frac{\pi}{2}$

$\therefore$ the x-intercept is $\frac{\pi}{2}$.

(ii) From Figure 2 (a), $f'(x) = 0$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

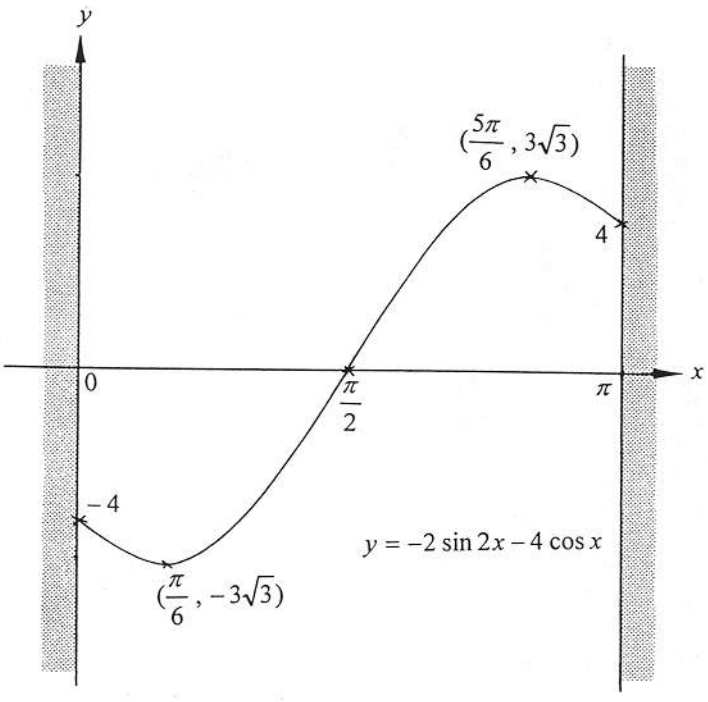
As $f'(x)$ changes from negative to positive as x increases through $\frac{\pi}{6}$,

so $(\frac{\pi}{6}, -3\sqrt{3})$ is a minimum point.

As $f'(x)$ changes from positive to negative as x increases through $\frac{5\pi}{6}$,

so $(\frac{5\pi}{6}, 3\sqrt{3})$ is a maximum point.

(c)



(d) $6 - 3\sqrt{3} \le g(x) \le 6 + 3\sqrt{3}$

Q.10 (a) $\overrightarrow{OC} = \dfrac{7\mathbf{a}+8\mathbf{b}}{15}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \dfrac{16}{21}\mathbf{b} - \mathbf{a}$

(b) (i) $\overrightarrow{OE} = r\overrightarrow{OC} = \dfrac{7r}{15}\mathbf{a} + \dfrac{8r}{15}\mathbf{b}$

(ii) $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \mathbf{a} + k\overrightarrow{AD}$

$= \mathbf{a} + k(\dfrac{16\mathbf{b}}{21} - \mathbf{a}) = (1-k)\mathbf{a} + \dfrac{16k}{21}\mathbf{b}$

Comparing the two expressions :

$$\begin{cases} \dfrac{7r}{15} = 1-k & ------(1) \\ \dfrac{8r}{15} = \dfrac{16}{21}k & ------(2) \end{cases}$$

$(1) \div (2)$: $\dfrac{7}{8} = \dfrac{21(1-k)}{16k}$

$14k = 21 - 21k$

$k = \dfrac{3}{5}$

Substitute $k = \dfrac{3}{5}$ into (1) : $\dfrac{7}{15}r = 1 - \dfrac{3}{5}$

$r = \dfrac{6}{7}$

$\therefore k = \dfrac{3}{5}$ and $r = \dfrac{6}{7}$.

(c) (i) Let $EC = x$.

Since $EC : ED = 1 : 2$, $ED = 2x$.

From (b), $\overrightarrow{OE} = \frac{6}{7}\overrightarrow{OC}$.

$\therefore$ $EO : EC = 6 : 1$, i.e. $EO = 6x$.

From (b), $\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AD}$.

$\therefore$ $EA : ED = 3 : 2$, i.e. $EA = 3x$.

$\therefore$ $EA : EO = 3x : 6x$

$= 1 : 2$.

(ii) In ΔEAC and ΔEOD,

$\angle AEC = \angle OED$

From (b), $\frac{EA}{EO} = \frac{1}{2} = \frac{EC}{ED}$

$\therefore \Delta EAC \sim \Delta EOD$.

$\angle EAC = \angle EOD$ (Corr $\angle$s of similar Δs)

$\therefore OACD$ is a cyclic quadrilateral.

(Converse of $\angle$s in the same segment)

Q.11 (a) $z_1 = 2(\cos 60° + i \sin 60°)$

$= 1 + \sqrt{3}i$

$z_3 = (\sqrt{3}i)\, z_1$

$= -3 + \sqrt{3}i$

(b) $\frac{z_2}{z_1} = \frac{z_1 + z_3}{z_1} = 1 + (\frac{z_3}{z_1})$

$= 1 + \sqrt{3}i \quad (\because\ z_3 = (\sqrt{3}i)z_1)$

$\angle AOB = \arg(\frac{z_2}{z_1}) = \arg(1 + \sqrt{3}i)$

$= 60°$

(c) (i) $\arg(z_3) = 150°$

$\arg(wz_3) = 150° + \theta$

$\arg(\bar{z}_3) = -150°$

If E represents the complex number $\bar{z}_3$,

$150° + \theta = -150° + 360°$

$\theta = 60°$

(ii) If E, O and A lie on a straight line,

$150° + \theta = 60° + 360\,k°$ or $150° + \theta = -120° + 360\,k°$
(k is an integer)

$\theta = 270°$ or $90°$.

Q.12 (a) S = Area of $ABCD$ – Area of ΔABE – Area of ΔCEF – Area of ΔADF

$$= 2(2k) - \frac{1}{2}(2)(2x) - \frac{1}{2}(x)(2k-2x) - \frac{1}{2}(2k)(2-x)$$
$$= x^2 - 2x + 2k$$

(b) (i) As E lies on BC, so $0 \le 2x \le 2k$, i.e. $0 \le x \le \frac{3}{2}$.

As F lies on CD, so $0 \le x \le 2$.

Combining the two inequalities, $0 \le x \le \frac{3}{2}$.

(ii) $S = x^2 - 2x + 2k$
$= x^2 - 2x + 3$
$= (x-1)^2 + 2$

As $x = 1$ lies in the range of possible value of x $(0 \le x \le \frac{3}{2})$,

$\therefore$ the least value of $S = 2$, which occurs when $x = 1$.

(iii) Since $S = x^2 - 2x + 3$ is a parabola and there is only a minimum in the range $0 \le x \le \frac{3}{2}$, so greatest value of S occurs at the end points.

At $x = 0, S = 3$.

At $x = \frac{3}{2}, S = (\frac{3}{2})^2 - 2(\frac{3}{2}) + 3 = \frac{9}{4}$.

$\therefore$ the greatest value of S is 3.

(c) (i) Put $k = \frac{3}{8}, S = x^2 - 2x + \frac{3}{4}$.

The range of possible values of x is $0 \le x \le \frac{3}{8}$.

As $x = 1$ does not lie in the above interval, the least value of S will not happen when $x = 1$.

$\therefore$ the student is incorrect.

(ii) As S is monotonic decreasing on $0 \le x \le \frac{3}{8}$,

least value of S occurs when $x = \frac{3}{8}$.

$\therefore$ least value of $S = (\frac{3}{8})^2 - 2(\frac{3}{8}) + \frac{3}{4} = \frac{9}{64}$

Q.13 (a) $h = \frac{V}{\pi x^2}$

$C = (2\pi xh) + k(\pi x^2)2$

$= 2\pi x(\frac{V}{\pi x^2}) + 2\pi x^2 k$

$= \frac{2V}{x} + 2\pi k x^2$

(b) $\frac{dC}{dx} = -\frac{2V}{x^2} + 4\pi k x$

$\frac{dC}{dx} = 0 \quad -\frac{2V}{x^2} + 4\pi k x = 0$

$x^3 = \frac{V}{2\pi k}$

$\frac{d^2C}{dx^2} = \frac{4V}{x^3} + 4\pi k$

Put $x^3 = (\frac{V}{2\pi k})$: $\frac{d^2C}{dx^2} = 12\pi k > 0.$

$\therefore$ C is a minimum.

$\frac{x}{h} = \frac{x}{V/\pi x^2} = \frac{\pi x^3}{V}$

$= \frac{\pi}{V}(\frac{V}{2\pi k}) = \frac{1}{2k}$

(c) (i) From (b), $x^3 = (\frac{V}{2\pi k}) = (\frac{256\pi}{2\pi(2)}) = 64$

$x = 4$

Since $\frac{x}{h} = \frac{1}{2k}$, $\frac{4}{h} = \frac{1}{2(2)}$

$h = 16$

(ii) Since $x^3 = \dfrac{V}{2\pi k}$, so x decreases when k increases.

As $h = \dfrac{V}{\pi x^2}$, so h increases when x decreases.

$\therefore$ the base radius of the can decreases and the height of the can increases.

(d) As the costs of the curved and plane surfaces remain unchanged, the ratio $\dfrac{x}{h} = \dfrac{1}{2k}$ is independent of the volume of the can.

$\therefore$ the ratio $\dfrac{\text{base radius}}{\text{height}}$ of the bigger can should remain identical to that of the smaller can in order to minimise the cost. So the worker is incorrect.

99-CE
A MATHS
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1999

ADDITIONAL MATHEMATICS PAPER 2

11.15 am – 1.15 pm (2 hours)
This paper must be answered in English

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2. All working must be clearly shown.
3. Unless otherwise specified, numerical answers must be **exact**.
4. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \, \mathrm{d}x$.

(3 marks)

2. Find $\int x(x+2)^{99} \, \mathrm{d}x$.

(4 marks)

3.

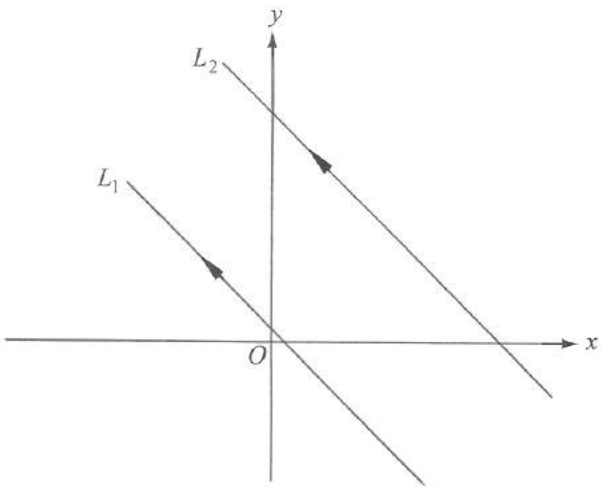

Figure 1

Figure 1 shows two parallel lines $L_1 : 2x + 2y - 1 = 0$ and $L_2 : 2x + 2y - 13 = 0$.

(a) Find the y-intercept of L_1.

(b) Find the distance between L_1 and L_2.

(c) L_3 is another line parallel to L_1. If the distance between L_1 and L_3 is equal to that between L_1 and L_2, find the equation of L_3.

(5 marks)

4.

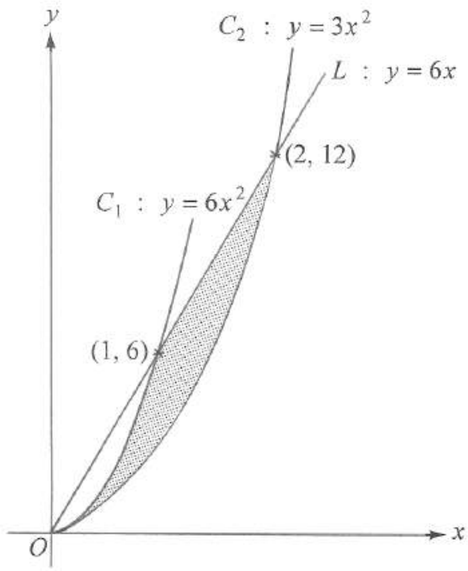

Figure 2

In Figure 2, the line $L : y=6x$ and the curves $C_1 : y=6x^2$ and $C_2 : y=3x^2$ all pass through the origin. L also intersects C_1 and C_2 at the points $(1, 6)$ and $(2, 12)$ respectively. Find the area of the shaded region.

(5 marks)

5. A family of straight lines is given by the equation

$$y-3+k(x-y+1)=0\ ,$$

where k is real.

(a) Find the equation of a line L_1 in the family whose x-intercept is 5.

(b) Find the equation of a line L_2 in the family which is parallel to the x-axis.

(c) Find the acute angle between L_1 and L_2.

(6 marks)

6. The slope at any point $(x,\ y)$ of a curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x + k.$$

If the curve **touches** the x-axis at the point (2, 0), find

(a) the value of k,

(b) the equation of the curve. (6 marks)

7. (a) Expand $(1+2x)^n$ in ascending powers of x up to the term x^3, where n is a positive integer.

(b) In the expansion of $(x-\frac{3}{x})^2(1+2x)^n$, the constant term is 210. Find the value of n.

(6 marks)

8. (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Find the general solution of the equation

$$\cos 6x + 4\cos 2x = 0.$$

(7 marks)

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Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9.

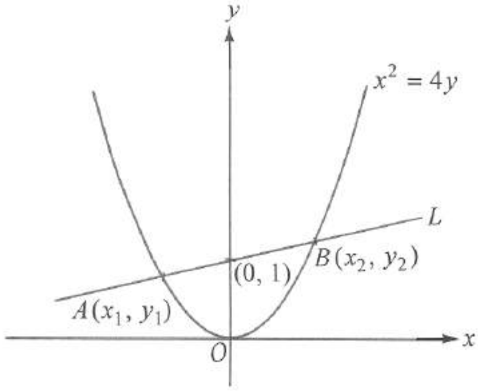


Figure 3

L is a straight line of slope m and passes through the point (0, 1). The line L cuts the parabola $x^2 = 4y$ at two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in Figure 3.

(a) Show that x_1 and x_2 are the roots of the equation

$$x^2 - 4mx - 4 = 0.$$ (3 marks)

(b) Find $(x_1 - x_2)^2$ in terms of m.

Hence, or otherwise, show that $AB = 4(1 + m^2)$. (6 marks)

(c) C is a circle with AB as a diameter.

(i) Find, in terms of m, the coordinates of the centre of C and its radius.

(ii) Find, in terms of m, the distance from the centre of C to the line $y + 1 = 0$.

State the geometrical relationship between C and the line $y + 1 = 0$. Explain your answer.

(7 marks)

10. $A(-3, 0)$ and $B(-1, 0)$ are two points and $P(x, y)$ is a variable point such that $PA = \sqrt{3}PB$. Let C be the locus of P.

(a) Show that the equation of C is $x^2 + y^2 = 3$. (3 marks)

(b) $T(a, b)$ is a point on C. Find the equation of the tangent to C at T. (2 marks)

(c) The tangent from A to C touches C at a point S in the second quadrant. Find the coordinates of S. (3 marks)

(d)

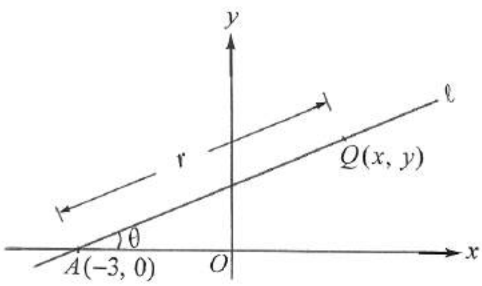


Figure 4

ℓ is a straight line which passes through point A and makes an angle θ with the positive x-axis, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $Q(x, y)$ is a point on ℓ such that $AQ = r$. (See Figure 4.)

(i) Write down the coordinates of Q in terms of r and θ.

(ii) ℓ cuts C at two distinct points H and K. Let $AH = r_1$, $AK = r_2$.

(1) Show that r_1 and r_2 are the roots of the quadratic equation $r^2 - 6r\cos\theta + 6 = 0$.

(2) Find the range of possible values of θ, giving your answers correct to three significant figures. (8 marks)

11.

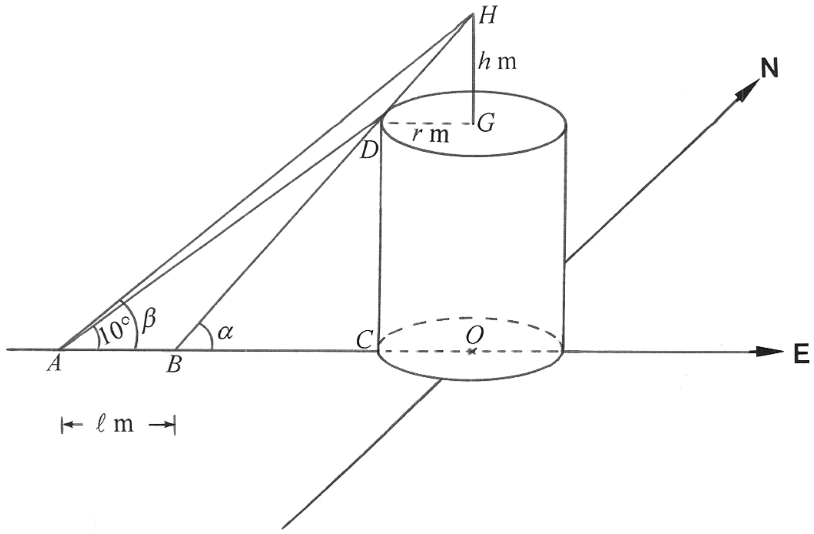


Figure 5

Figure 5 shows a right cylindrical tower with a radius of r m standing on horizontal ground. A vertical pole HG, h m in height, stands at the centre G of the roof of the tower. Let O be the centre of the base of the tower. C is a point on the circumference of the base of the tower due west of O and D is a point on the roof vertically above C. A man stands at a point A due west of O. The angles of elevation of D and H from A are 10° and β respectively. The man walks towards the east to a point B where he can just see the top of the pole H as shown in Figure 5. (Note : If he moves forward, he can no longer see the pole.) The angle of elevation of H from B is α. Let $AB = \ell$ m.

(a) Show that $AD = \dfrac{\ell \sin \alpha}{\sin(\alpha - 10°)}$ m.

Hence (i) express CD in terms of ℓ and α,

(ii) show that $h = \dfrac{\ell \sin^2 \alpha \sin(\beta - 10°)}{\sin(\alpha - 10°)\sin(\alpha - \beta)}$.

(Hint : You may consider ΔADH.)

(6 marks)

(b) **In this part, numerical answers should be given correct to two significant figures.**

Suppose $\alpha = 15°$, $\beta = 10.2°$ and $\ell = 97$.

(i) Find

(1) the height of the pole HG,

(2) the height and radius of the tower.

(ii) P is a point south-west of O. Another man standing at P can just see the top of the pole H. Find

(1) the distance of P from O,

(2) the bearing of B from P.

(10 marks)

Go on to the next page

12. (a) Prove, by mathematical induction, that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta},$$

where $\sin\theta \neq 0$, for all positive integers n.

(6 marks)

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x},$$

where $\cos x \neq 0$.

(2 marks)

(c) Using (a) and (b), evaluate

$$\int_{0.1}^{0.5} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} \right)^2 \mathrm{d}x,$$

giving your answer correct to two significant figures.

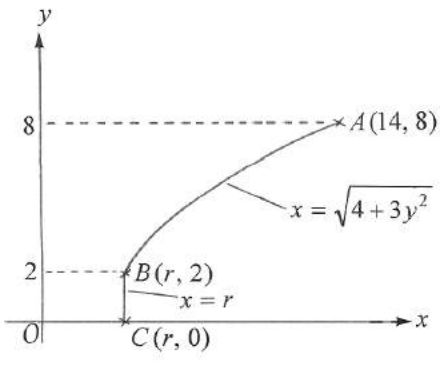
(4 marks)

(d) Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x)\,\mathrm{d}x \circ$$

(4 marks)

13.



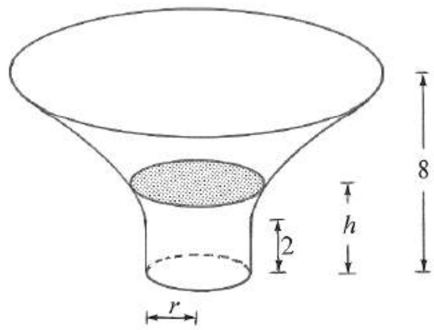


Figure 6(a) **Figure 6(b)**

A curve passes through three points $A(14, 8)$, $B(r, 2)$ and $C(r, 0)$ as shown in Figure 6(a). The curve consists of two parts. The equation of the part joining A and B is $x = \sqrt{4+3y^2}$ and the part joining B and C is the vertical line $x = r$.

(a) Find the value of r.

(2 marks)

(b) A pot, 8 units in height, is formed by revolving the curve and the line segment OC about the y-axis, where O is the origin. (See Figure 6(b).) If the pot contains water to a depth of h units, where $h > 2$, show that the volume of water V in the pot is $(h^3 + 4h + 16)\pi$ cubic units.

(7 marks)

(c) Initially, the pot in (b) contains water to a depth greater than 3 units. The water is now pumped out at a constant rate of 2π cubic units per second. Find the rate of change of the depth of the water in the pot with respect to time when

(i) the depth of the water is 3 units, and

(ii) the depth of the water is 1 unit.

(7 marks)

END OF PAPER

Outlines of Solutions

1999 Additional Mathematics

Paper 2

Section A

1. $\frac{\pi}{4}$

2. $\frac{(x+2)^{101}}{101} - \frac{(x+2)^{100}}{50} + c$, where c is a constant

3. (a) $\frac{1}{2}$

   (b) $3\sqrt{2}$

   (c) $2x + 2y + 11 = 0$

4. 3

5. (a) $x + y - 5 = 0$

   (b) $y - 3 = 0$

   (c) $\frac{\pi}{4}$

6. (a) –8

   (b) $y = x^3 - x^2 - 8x + 12$

7. (a) $1 + 2{}_nC_1 x + 4{}_nC_2 x^2 + 8{}_nC_3 x^3 + \ldots$

   (b) 4

8. (b) $k\pi \pm \frac{\pi}{4}$, where k is an integer

Section B

Q.9 (a) The equation of L is $y = mx + 1$.

Substitute $y = mx + 1$ into $x^2 = 4y$:

$x^2 = 4(mx + 1)$

$x^2 - 4mx - 4 = 0$

$\therefore x_1, x_2$ are the roots of the equation $x^2 - 4mx - 4 = 0$.

(b) $\begin{cases} x_1 + x_2 = 4m \\ x_1 x_2 = -4 \end{cases}$

$$\begin{aligned}(x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= (4m)^2 - 4(-4) \\ &= 16(m^2 + 1)\end{aligned}$$

$$\begin{aligned}AB^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (x_1 - x_2)^2 + (mx_1 + 1 - mx_2 - 1)^2 \\ &= (x_1 - x_2)^2 + (mx_1 - mx_2)^2 \\ &= (1 + m^2)\,[16(m^2 + 1)]\end{aligned}$$

$AB = 4(1 + m^2)$

(c) (i) x-coordinate of centre of $C = \dfrac{x_1 + x_2}{2} = 2m$

y-coordinate of centre of $C = \dfrac{y_1 + y_2}{2} = \dfrac{mx_1 + 1 + mx_2 + 1}{2}$

$= \dfrac{m}{2}(4m) + 1 = 2m^2 + 1$

$\therefore$ the coordinates of the centre are $(2m, 2m^2 + 1)$.

Radius of $C = \dfrac{AB}{2} = 2(1 + m^2)$

(ii) Distance from centre of C to $y + 1 = 0$

$= |\,2m^2 + 1 - (-1)\,|$

$= 2\,(m^2 + 1)$

As the distance from centre of C to $y + 1 = 0$ is equal to the radius C, the line $y + 1 = 0$ is a tangent to C.

Q.10 (a) $PA=\sqrt{3}PB$

$\sqrt{(x+3)^2+y^2}=\sqrt{3}\sqrt{(x+1)^2+y^2}$

$x^2+6x+9+y^2=3(x^2+2x+1+y^2)$

$x^2+y^2=3$

(b) Differentiate $x^2+y^2=3$ with respect to x :

$2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$

Equation of tangent at $T(a,b)$ is

$\frac{y-b}{x-a}=\frac{-a}{b}$

$ax+by=a^2+b^2$

(c) Substitute $A(-3, 0)$ into the equation of tangent :

$a(-3)+b(0)=3$

$a=-1$

$b=\sqrt{3-(-1)^2}$ (Θ S lies in the 2nd quadrant.)

$=\sqrt{2}$

$\therefore$ the coordinates of S are $(-1,\sqrt{2})$.

(d) (i) The coordinates of Q are $(-3+r\cos\theta, r\sin\theta)$.

(ii) (1) Substitute $(-3+r\cos\theta, r\sin\theta)$ into C :

$(-3+r\cos\theta)^2+(r\sin\theta)^2=3$

$9-6r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta=3$

$r^2-6r\cos\theta+6=0$ ---- (*)

Since $AH=r_1$, $AK=r_2$, r_1 and r_2 are the roots of (*).

(2) Since ℓ cuts C at two distinct points, (*) has two distinct real roots.

$(6\cos\theta)^2 - 4(6) > 0$

$\cos^2\theta > \frac{2}{3}$

$\cos\theta > \sqrt{\frac{2}{3}}$ or $\cos\theta < -\sqrt{\frac{2}{3}}$ (rejected)

$\therefore -0.615 < \theta < 0.615$ (correct to 3 sig. figures)

Q.11 (a) Consider ΔABD :

By Sine Law,

$$\frac{AD}{\sin\angle ABD}=\frac{\ell}{\sin\angle ADB}$$

$$\frac{AD}{\sin(180^\circ-\alpha)}=\frac{\ell}{\sin(\alpha-10^\circ)}$$

$$AD=\frac{\ell\sin\alpha}{\sin(\alpha-10^\circ)}\text{ m}$$

(i) Consider ΔACD :

$$CD = AD\sin 10^\circ$$

$$=\frac{\ell\sin\alpha\sin 10^\circ}{\sin(\alpha-10^\circ)}\text{ m}$$

(ii) Consider ΔADH :

$$\frac{AD}{\sin(\alpha-\beta)}=\frac{DH}{\sin(\beta-10^\circ)}$$

$$DH = AD\frac{\sin(\beta-10^\circ)}{\sin(\alpha-\beta)}=\frac{\ell\sin\alpha\sin(\beta-10^\circ)}{\sin(\alpha-10^\circ)\sin(\alpha-\beta)}$$

Consider ΔDHG :

$$h = DH\sin\alpha$$

$$=\frac{\ell\sin^2\alpha\sin(\beta-10^\circ)}{\sin(\alpha-10^\circ)\sin(\alpha-\beta)}$$

(b) (i) (1) Using (a) (ii) :

$$\text{height of pole }=\frac{97\sin^2 15^\circ\sin(10.2^\circ-10^\circ)}{\sin(15^\circ-10^\circ)\sin(15^\circ-10.2^\circ)}$$

$$= 3.1100 = 3.1\text{ m (correct to 2 sig. fig.)}$$

(2) Using (a) (i) :

$$\text{height of tower }\ CD=\frac{97\sin 15^\circ\sin 10^\circ}{\sin(15^\circ-10^\circ)}$$

$$= 50.020 = 50\text{ m (correct to 2 sig. fig.)}$$

$$\text{radius of tower}=\frac{h}{\tan 15^\circ}$$

$$=\frac{3.1100}{\tan 15^\circ}$$

$$= 11.607 = 12\text{ m (correct to 2 sig. fig.)}$$

(ii) (1) Consider ΔHPO :

$$\tan \angle HPO = \frac{OH}{OP}$$

$$OP = \frac{OH}{\tan 15^\circ}$$

$$= \frac{3.1100 + 50.020}{\tan 15^\circ}$$

$$= 198.28$$

= 200 m (correct to 2 sig. fig.)

(2) $\angle BPO = \frac{1}{2}(180^\circ - 45^\circ) \quad = 67.5^\circ$

Bearing of B from P is N(67. 5° – 45°)W, i.e. N22. 5° W.

Q.12 (a) For $n=1$, LHS $=\cos\theta$.

$$\text{RHS}=\frac{\sin 2\theta}{2\sin\theta}$$

$$=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta=\text{LHS}.$$

$\therefore$ the statement is true for $n = 1$.

Assume $\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2k-1)\theta=\frac{\sin 2k\theta}{2\sin\theta}$

for some positive integer k.

Then $\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2k-1)\theta+\cos[2(k+1)-1]\theta$

$$=\frac{\sin 2k\theta}{2\sin\theta}+\cos(2k+1)\theta$$

$$=\frac{\sin 2k\theta+2\sin\theta\cos(2k+1)\theta}{2\sin\theta}$$

$$=\frac{\sin 2k\theta+\sin(2k+2)\theta-\sin 2k\theta}{2\sin\theta}$$

$$=\frac{\sin 2(k+1)\theta}{2\sin\theta}$$

The statement is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, the statement is true for all positive integers n.

(b) Using (a) : $\cos\theta+\cos 3\theta+\cos 5\theta=\frac{\sin 6\theta}{2\sin\theta}$, where $\sin\theta\neq 0$.

Put $\theta=\frac{\pi}{2}-x$:

$$\cos(\frac{\pi}{2}-x)+\cos 3(\frac{\pi}{2}-x)+\cos 5(\frac{\pi}{2}-x)=\frac{\sin 6(\frac{\pi}{2}-x)}{2\sin(\frac{\pi}{2}-x)}$$

$$\cos(\frac{\pi}{2}-x)+\cos(\frac{3\pi}{2}-3x)+\cos(\frac{5\pi}{2}-5x)=\frac{\sin(3\pi-6x)}{2\sin(\frac{\pi}{2}-x)}$$

$$\sin x-\sin 3x+\sin 5x=\frac{\sin 6x}{2\cos x}$$, where $\sin(\frac{\pi}{2}-x)=\cos x\neq 0$.

(c) $\int_{0.1}^{0.5} (\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x})^2 \, dx$

$= \int_{0.1}^{0.5} [\frac{\sin 6x}{2\cos x} / \frac{\sin 6x}{2\sin x}]^2 \, dx$

$= \int_{0.1}^{0.5} \tan^2 x \, dx$

$= \int_{0.1}^{0.5} (\sec^2 x - 1) \, dx$

$= [\tan x - x]_{0.1}^{0.5}$

$= 0.046$ (correct to 2 sig. fig.)

(d) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x) dx$

$= [-\cos x - \cos 3x - \cos 5x - \cos 7x - \cdots - \cos 1999x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -\frac{1}{2}[\frac{\sin 2000x}{\sin x}]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{1}{2}$

Q.13 (a) Substitute $(r, 2)$ into $x=\sqrt{4+3y^2}$:

$r=\sqrt{4+3(2)^2}=4$

(b) V = Volume of lower cylindrical part + volume of upper part

Volume of lower cylindrical part $=\pi r^2 h$

$=\pi(4)^2(2)$

$=32\pi$

Volume of upper part $=\pi\int_2^h x^2\,\mathrm{d}y$

$=\pi\int_2^h (4+3y^2)\,\mathrm{d}y$

$=\pi[4y+y^3]_2^h$

$=(h^3+4h-16)\pi$

$\therefore\ V=32\pi+(h^3+4h-16)\pi$

$=(h^3+4h+16)\pi$ cubic units

(c) (i) Let h units be the depth of water at time t.

$\frac{\mathrm{d}V}{\mathrm{d}t}=\pi(3h^2+4)\frac{\mathrm{d}h}{\mathrm{d}t}$

Put $\frac{\mathrm{d}V}{\mathrm{d}t}=-2\pi$ and $h=3$:

$-2\pi=\pi[3(3)^2+4]\frac{\mathrm{d}h}{\mathrm{d}t}$

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-2}{31}$ units per sec.

$\therefore$ the depth decreases at a rate $\frac{2}{31}$ units per sec.

(ii) When $h = 1$, the water remained is in the cylindrical part only.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\text{base area of cylilnder}}$$

$$= \frac{-2\pi}{\pi(4)^2}$$

$$= -\frac{1}{8} \text{ units per sec.}$$

$\therefore$ the depth decreases at a rate $\frac{1}{8}$ units per sec.