Section A (42 marks)

Answer **ALL** questions in this section.

1. Let $f(x) = x^2 + (1 - m)x + 2m - 5$, where m is a constant. Find the discriminant of the equation $f(x) = 0$.

   Hence find the range of values of m so that $f(x) > 0$ for all real values of x.

   (5 marks)

2. Let $z = -1 + \sqrt{3}i$. Express z in polar form.

   Hence find $z^5 + \bar{z}^5$.

   (5 marks)

3. Using the information in the following table, sketch the graph of $y = f(x)$, where $f(x)$ is a polynomial.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f(x)$		1		2		1	
$f'(x)$	< 0	0	> 0	0	< 0	0	> 0

   (5 marks)

4. By considering the two cases $x > 0$ and $x < 0$, or otherwise, solve the inequality

$$x - \frac{5}{x} > 4.$$

   (6 marks)

5. In the same Argand diagram, sketch the locus of the point representing the complex number z in each of the following cases :

   (a) $|z - (3 + i)| = 3$,

   (b) $|z| = |z - 8i|$.

   Hence, or otherwise, find the complex number(s) represented by the point(s) of intersection of the two loci.

   (6 marks)

6. $P(4, 1)$ is a point on the curve $y^2 + y\sqrt{x} = 3$, where $x > 0$.

   (a) Find the value of $\frac{dy}{dx}$ at P.

   (b) Find the equation of the normal to the curve at P.

   (7 marks)

7. Let $\overrightarrow{OP} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OQ} = -6\mathbf{i} + 4\mathbf{j}$. Let R be a point on PQ such that $PR : RQ = k : 1$, where $k > 0$.

   (a) Express $\overrightarrow{OR}$ in terms of k, $\mathbf{i}$ and $\mathbf{j}$.

   (b) Express $\overrightarrow{OP} \cdot \overrightarrow{OR}$ and $\overrightarrow{OQ} \cdot \overrightarrow{OR}$ in terms of k.

   (c) Find the value of k such that OR bisects $\angle POQ$.

   (8 marks)

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Section B (48 marks)

Answer any **THREE** questions in this section.
Each question carries 16 marks.

8.

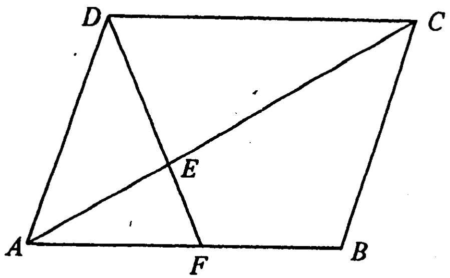

Figure 1

In Figure 1, $ABCD$ is a parallelogram and F is a point on AB. DF meets AC at a point E such that $DE : EF = \lambda : 1$, where λ is a positive number. Let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$ and $\overrightarrow{AE} = h\overrightarrow{AC}$, $\overrightarrow{AF} = k\overrightarrow{AB}$, where h, k are positive numbers.

(a) (i) Express $\overrightarrow{AE}$ in terms of h, $\mathbf{p}$ and $\mathbf{q}$.

(ii) Express $\overrightarrow{AE}$ in terms of λ, k, $\mathbf{p}$ and $\mathbf{q}$.

Hence show that $\lambda = \frac{1}{k}$. (5 marks)

(b) It is given that $|\mathbf{p}| = 3$, $|\mathbf{q}| = 2$, $\angle DAB = \frac{\pi}{3}$.

(i) Find $\mathbf{p} \cdot \mathbf{q}$.

(ii) Suppose DF is perpendicular to AC.

(1) By expressing $\overrightarrow{DF}$ in terms of k, $\mathbf{p}$ and $\mathbf{q}$, find the value of k.

(2) Using (a), or otherwise, find the length of AE.

(11 marks)

9.

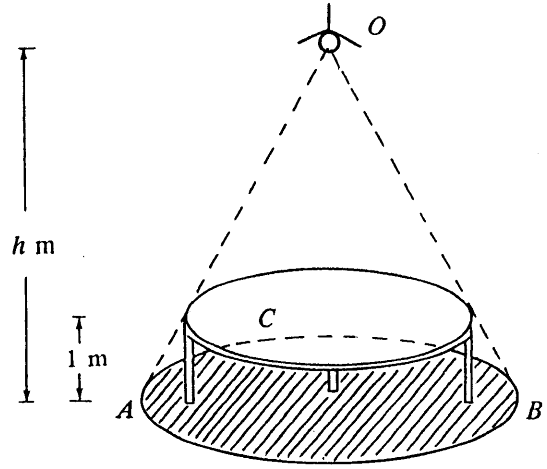

Figure 2

A small lamp O is placed h m above the ground, where $1 < h \leq 5$. Vertically below the lamp is the centre of a round table of radius 2 m and height 1 m. The lamp casts a shadow ABC of the table on the ground (see Figure 2). Let S m^2 be the area of the shadow.

(a) Show that $S = \frac{4\pi h^2}{(h-1)^2}$. (3 marks)

(b) If the lamp is lowered vertically at a constant rate of $\frac{1}{8}$ m s^{-1}, find the rate of change of S with respect to time when $h = 2$. (5 marks)

(c) Let V m^3 be the volume of the cone $OABC$.

(i) Show that $V = \frac{4\pi h^3}{3(h-1)^2}$.

(ii) Find the minimum value of V as h varies.

Does S attain a minimum when V attains its minimum? Explain your answer. (8 marks)

10. Let $f(x) = 12x^2 + 2px - q$

and $g(x) = 12x^2 + 2qx - p$,

where p, q are distinct real numbers. α , β are the roots of the equation $f(x) = 0$ and α, γ are the roots of the equation $g(x) = 0$.

(a) Using the fact that $f(\alpha) = g(\alpha)$, find the value of α.

Hence show that $p + q = 3$.

(3 marks)

(b) Express β and γ in terms of p.

(4 marks)

(c) Suppose $|\beta^3 + \gamma^3| < \frac{7}{24}$.

(i) Find the range of possible values of p.

(ii) Furthermore, if $p > q$, write down the possible integral values of p and q.

(9 marks)

11. (a) Let α, β be the roots of the equation

$$x^2 - x + 1 = 0 \text{ .... } (*),$$

where $-\pi < \arg \beta < \arg \alpha < \pi$.

Express α and β in polar form. (3 marks)

(b)

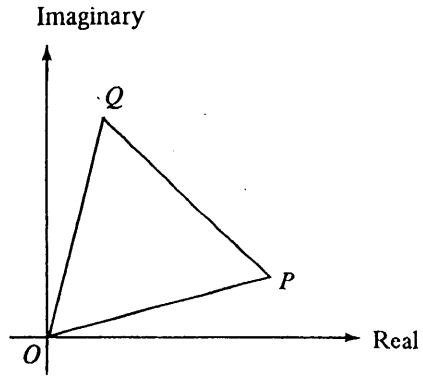

Figure 3

Figure 3 shows an Argand diagram in which OPQ is an equilateral triangle. The complex numbers represented by O, P and Q are 0, z_1 and z_2 respectively.

(i) Find the values of $\left|\frac{z_2}{z_1}\right|$ and $\arg\left(\frac{z_2}{z_1}\right)$.

Hence show that $\frac{z_2}{z_1}$ is a root of the equation (*) in (a).

(ii) Using (i), or otherwise, show that

$$z_1^2 + z_2^2 = z_1 z_2.$$

(iii) It is given that $|z_1| = 2$. Find

(1) $|z_1 + z_2|$,

(2) $|z_1^2 + z_2^2|$.

(13 marks)

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12.

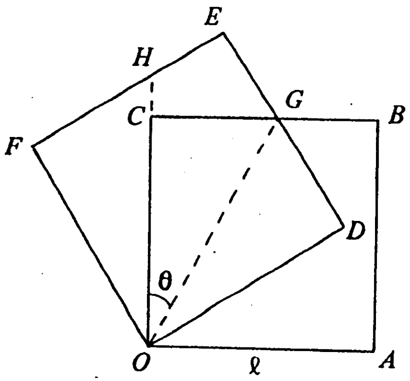

Figure 4

In Figure 4, $OABC$ is the position of a square of side ℓ. The square is rotated anticlockwise about O to a new position $ODEF$. BC cuts DE at G and OC produced cuts EF at H. Let $\angle COG = \theta$, where $\frac{\pi}{8} < \theta < \frac{\pi}{4}$.

(a) Name a triangle which is congruent to ΔOCG.

Hence show that the area of ΔOFH is $\frac{\ell^2}{2\tan 2\theta}$. (3 marks)

(b) Let S be the sum of the areas of ΔOFH and the quadrilateral $ODGC$.

(i) Show that $S = \frac{\ell^2}{2}\left(\frac{2 - \cos 2\theta}{\sin 2\theta}\right)$.

(ii) Find the range of values of θ for which S is

(1) increasing,

(2) decreasing.

Hence find the minimum value of S. (11 marks)

(c) Find the maximum value of the area of the quadrilateral $CGEH$. (2 marks)

END OF PAPER

Section A (42 marks)

Answer **ALL** questions in this section.

1. The slope at any point (x, y) of a curve C is given by

$$\frac{dy}{dx} = 2x\sqrt{x^2 + 1}$$

and C cuts the y-axis at the point $(0, 1)$. Find the equation of C.

(Hint : Let $u = x^2 + 1$.)

(5 marks)

2. A family of straight lines is given by the equation

$$2x - 3y + 4 + k(4x + 2y - 1) = 0,$$

where k is any constant.

(a) Find the equation of the line in the family which passes through the point $(1, 0)$.

(b) Find the equation of the line in the family with slope 2.

(6 marks)

3. $A(t, \frac{1}{2}t^2)$ is a point on the parabola $x^2 = 2y$. B is the point $(2, 2)$.

(a) Find the equation of the locus of the mid-point of AB as A moves along the parabola.

(b) Sketch the locus in (a).

(6 marks)

4. Given $(x^2 + \frac{1}{x})^5 - (x^2 - \frac{1}{x})^5 = ax^7 + bx + \frac{c}{x^5}$, find the values of a, b and c.

Hence evaluate $(2 + \frac{1}{\sqrt{2}})^5 - (2 - \frac{1}{\sqrt{2}})^5$.

(6 marks)

5.

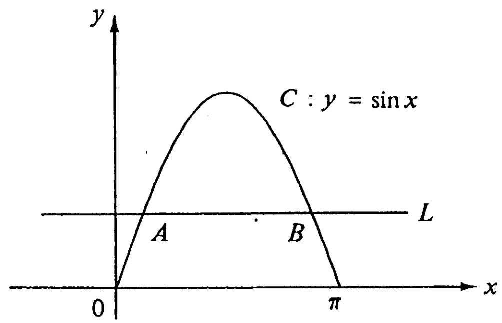

Figure 1

Figure 1 shows the curve $C : y = \sin x$ for $0 \le x \le \pi$.

(a) Find the area of the finite region bounded by the curve C and the x-axis.

(b) A horizontal line L cuts C at two points A and B. A is the point $(\frac{\pi}{6}, \frac{1}{2})$.

(i) Write down the coordinates of B.

(ii) Find the area of the finite region bounded by C and L.

(6 marks)

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6. Prove, by mathematical induction, that $(8^n - 1)$ is divisible by 7 for all positive integers n.

(6 marks)

7.

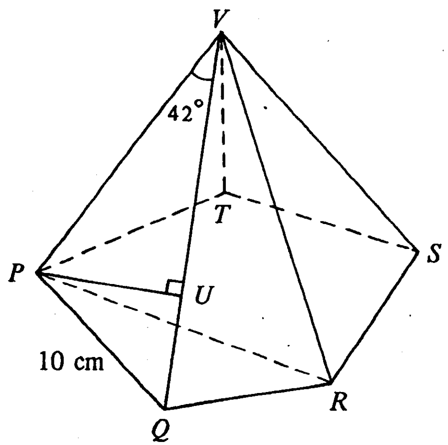

Figure 2

In Figure 2, *VPQRST* is a right pyramid whose base *PQRST* is a regular pentagon. $PQ = 10$ cm and $\angle PVQ = 42°$. U is a point on VQ such that PU is perpendicular to VQ. Find, correct to 3 significant figures,

(a) PU and PR,

(b) the angle between the faces VPQ and VQR.

(7 marks)

Section B (48 marks)

Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. Let n be an integer greater than 1.

(a) Show that

$$\frac{d}{dx}[x^{n-1}(1-x^2)^{\frac{3}{2}}] = (n-1)x^{n-2}\sqrt{1-x^2} - (n+2)x^n\sqrt{1-x^2}.$$

(4 marks)

(b) Using (a), show that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \frac{n-1}{n+2}\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx.$$

(3 marks)

(c) Using the substitution $x = \sin\theta$, evaluate

$$\int_0^1 \sqrt{1-x^2}\,dx.$$

(3 marks)

(d) Using (b) and (c), evaluate the following integrals :

(i) $\int_0^1 x^4\sqrt{1-x^2}\,dx$,

(ii) $\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\,d\theta$.

(6 marks)

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9. (a) Show that $\cos^2 A - \cos^2 B = \sin(A + B)\sin(B - A)$.

(3 marks)

(b) ABC is a triangle.

(i) Using (a), show that

$$\cos^2 A - \cos^2 B + \sin^2 C = 2\cos A \sin B \sin C.$$

(ii) If $\cos^2 A - \cos^2 B - \cos^2 C = -1$, show that ABC is a right-angled triangle.

(6 marks)

(c) Using (a), or otherwise, show that

$$\cos^2 x - \sin^2 y = \cos(x + y)\cos(x - y).$$

Hence find the general solution of

$$\cos^2 2\theta - \sin^2 3\theta + \cos\theta \sin 5\theta = 0.$$

(7 marks)

10.

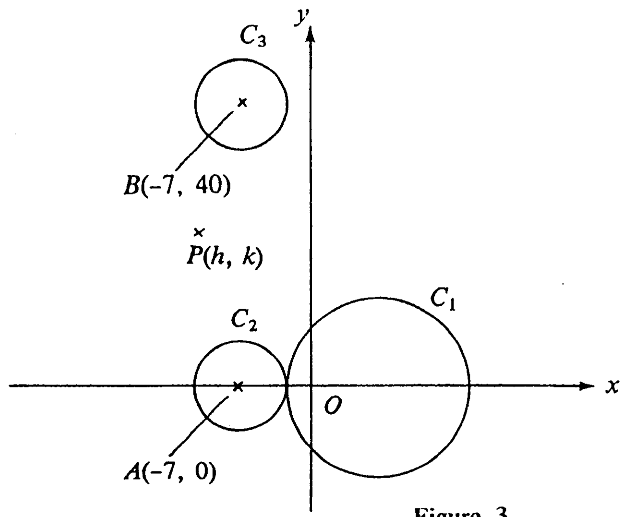

Figure 3

C_1 is the circle $x^2 + y^2 - 16x - 36 = 0$ and C_2 is a circle centred at the point $A(-7, 0)$. C_1 and C_2 touch externally as shown in Figure 3. $P(h, k)$ is a point in the second quadrant.

(a) Find the centre and radius of C_1.

Hence find the radius of C_2. (4 marks)

(b) If P is the centre of a circle which touches both C_1 and C_2 externally, show that

$$8h^2 - k^2 - 8h - 48 = 0.$$ (5 marks)

(c) C_3 is a circle centred at the point $B(-7, 40)$ and of the same radius as C_2.

(i) If P is the centre of a circle which touches both C_2 and C_3 externally, write down the equation of the locus of P.

(ii) Find the equation of the circle, with centre P, which touches all the three circles C_1, C_2 and C_3 externally.

(7 marks)

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11. (a)

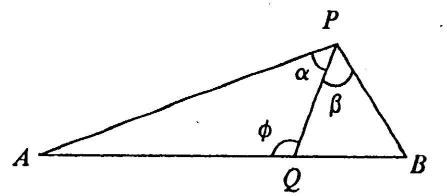

Figure 4(a)

Figure 4(a) shows a triangle PAB and Q is a point on AB. Let $\angle AQP = \phi$, $\angle APQ = \alpha$ and $\angle QPB = \beta$.

(i) Express $\dfrac{QA}{PA}$ in terms of α and ϕ.

(ii) If $\dfrac{QA}{PA} = \dfrac{QB}{PB}$, show that $\alpha = \beta$.

(3 marks)

(b)

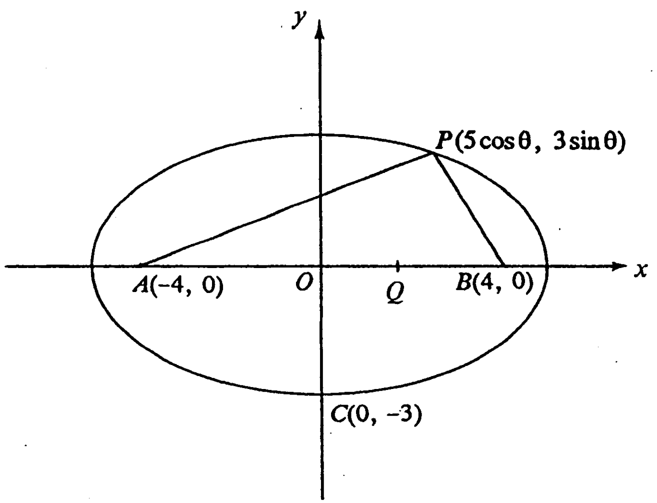

Figure 4(b)

In Figure 4(b), $P(5\cos\theta,\ 3\sin\theta)$ is a point on the ellipse $9x^2 + 25y^2 = 225$, where $\sin\theta \neq 0$. A and B are the points $(-4, 0)$ and $(4, 0)$ respectively. Q is a point on AB such that the normal to the ellipse at P passes through Q.

(i) Show that the equation of PQ is

$$5x\sin\theta - 3y\cos\theta - 16\sin\theta\cos\theta = 0.$$

Hence find the coordinates of Q in terms of θ.

(ii) Show that $PA = 5 + 4\cos\theta$.

Hence, using (a) (ii), show that PQ bisects $\angle APB$.

(13 marks)

Go on to the next page

12. (a) Using the substitution $y - k = \cos\theta$, show that

$$\int_{k-1}^{k+1} \sqrt{1 - (y-k)^2}\, dy = \frac{\pi}{2}.$$

Hence show that

(i) $$\int_0^2 \left[2 + \sqrt{1 - (y-1)^2}\right]^2 dy = \frac{28}{3} + 2\pi,$$

(ii) $$\int_2^4 \left[2 - \sqrt{1 - (y-3)^2}\right]^2 dy = \frac{28}{3} - 2\pi.$$

(8 marks)

(b)

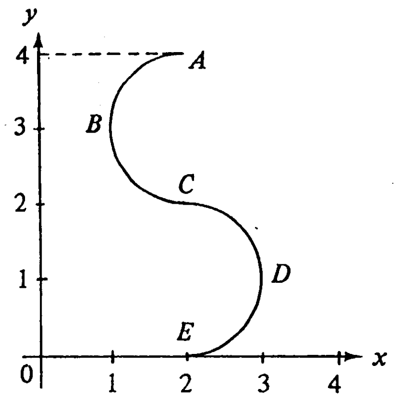

Figure 5(a)

Figure 5(a) shows two semicircles ABC and CDE centred at (2, 3) and (2, 1) respectively. Their radii are both equal to 1.

(i) Show that the equation of the semicircle ABC is

$$x = 2 - \sqrt{1 - (y-3)^2},$$

and that of the semicircle CDE is

$$x = 2 + \sqrt{1 - (y-1)^2}.$$

(ii) A pot is formed by revolving the curve $ABCDE$ and the line segment OE about the y-axis, where O is the origin. Using (a), find the capacity of the pot.

(5 marks)

(c)

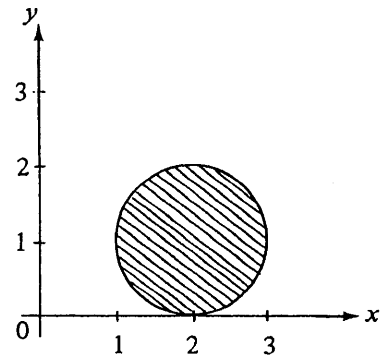

Figure 5(b)

The shaded region enclosed by the circle $(x-2)^2 + (y-1)^2 = 1$, as shown in Figure 5(b), is revolved about the y-axis to form a solid. Using (a) and (b), or otherwise, find the volume of the solid.

(3 marks)

END OF PAPER