91-CE
A MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1991

ADDITIONAL MATHEMATICS PAPER I

8.30 am-10.30 am (2 hours)
This paper must be answered in English

Answer ALL questions in Section A and any THREE questions in Section B.

All working must be clearly shown.

Unless otherwise specified in a question, numerical answers must be given in **exact value.**

SECTION A (42 marks)

Answer **ALL** questions in this section.

1. In an Argand diagram, sketch the locus of the point representing the complex number z which satisfies the equation

$$|z - 3i| = 2.$$

Let P be the point representing the complex number $4 + 3i$. Write down the complex number represented by the point on the locus which is nearest to P.

(4 marks)

2. Let $f(x) = \dfrac{1}{1 + x}$.

Find $f'(x)$ from first principles.

(5 marks)

3. Solve $|x - 2| = |x^2 - 4|$.

(5 marks)

4. Let $y = x + \sin 2x$, where $0 \leq x \leq \pi$.

Find (a) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$,

(b) the maximum and minimum values of y.

(7 marks)

5.

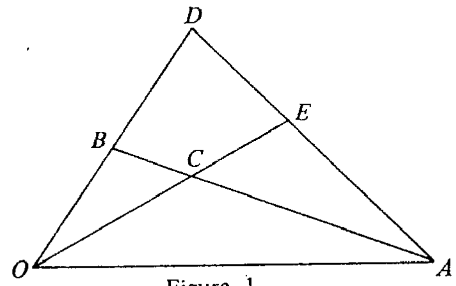

Figure 1

In Figure 1, OAD is a triangle and B is the mid-point of OD. The line OE cuts the line AB at C such that $AC : CB = 3 : 1$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) (i) Let $OC : CE = k : 1$. Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

(ii) Let $AE : ED = m : 1$. Express $\overrightarrow{OE}$ in terms of m, $\mathbf{a}$ and $\mathbf{b}$.

Hence find k and m.

(7 marks)

6. Let C be the curve $y = \frac{1}{x} + x$, where $x \neq 0$. $P(1, 2)$ and $Q(\frac{1}{2}, \frac{5}{2})$ are two points on C.

(a) Find equations of the tangent and normal to C at P.

(b) Show that the tangent to C at Q passes through the point $A(0, 4)$.

(7 marks)

7. p, q and k are real numbers satisfying the following conditions :

$$\begin{cases} p + q + k = 2, \\ pq + qk + kp = 1. \end{cases}$$

(a) Express pq in terms of k.

(b) Find a quadratic equation, with coefficients in terms of k, whose roots are p and q.

Hence find the range of possible values of k.

(7 marks)

SECTION B (48 marks)

Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. A, B and C are three points on a plane such that

$$\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j},$$

$$\overrightarrow{BC} = 7\mathbf{i} + \mathbf{j},$$

and $$\overrightarrow{OC} = x\mathbf{i} + y\mathbf{j},$$

where O is the origin.

(a) Find $\overrightarrow{CA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ in terms of x, y, $\mathbf{i}$ and $\mathbf{j}$.

(4 marks)

(b) Given $\overrightarrow{AB} \cdot \overrightarrow{BC} = 4\, \overrightarrow{BC} \cdot \overrightarrow{CA}$.

(i) Show that $y = 30 - 7x$.

(ii) If $|\overrightarrow{BC}| = \sqrt{5}\, |\overrightarrow{CA}|$ and x, y are positive,

(1) find x and y,

(2) show that CA is perpendicular to AB,

(3) show that O lies on AB.

(12 marks)

9. Let $f(x) = x^2 + 2x - 2$

and $g(x) = -2x^2 - 12x - 23$.

(a) Express $g(x)$ in the form $a(x+b)^2 + c$, where a, b and c are real constants.

Hence show that $g(x) < 0$ for all real values of x.

(3 marks)

(b) Let k_1 and k_2 $(k_1 > k_2)$ be the two values of k such that the equation $f(x) + kg(x) = 0$ has equal roots.

(i) Find k_1 and k_2.

(ii) Show that

$$f(x) + k_1 g(x) \leq 0$$

and $f(x) + k_2 g(x) \geq 0$ for all real values of x.

(8 marks)

(c) Using (a) and (b), or otherwise, find the greatest and least values of $\frac{f(x)}{g(x)}$.

(5 marks)

10. (a) (i) Solve $t^2 + t + 1 = 0$, expressing the roots in the form $\cos\theta + i\sin\theta$, where $-\pi < \theta \leq \pi$.

(ii) Hence find the roots of $z^6 + z^3 + 1 = 0$ in the form $\cos\theta + i\sin\theta$, where $-\pi < \theta \leq \pi$.

(7 marks)

(b) (i) Show that

$$[z - \cos\theta - i\sin\theta][z - \cos(-\theta) - i\sin(-\theta)] = z^2 - 2z\cos\theta + 1.$$

(ii) Hence show that

$$z^6 + z^3 + 1 = (z^2 - 2z\cos\frac{2\pi}{9} + 1)(z^2 - 2z\cos\frac{4\pi}{9} + 1)(z^2 - 2z\cos\frac{8\pi}{9} + 1) \ldots\ldots(*).$$

(5 marks)

(c) By substituting a suitable value of z into $(*)$, show that

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}.$$

(4 marks)

11.

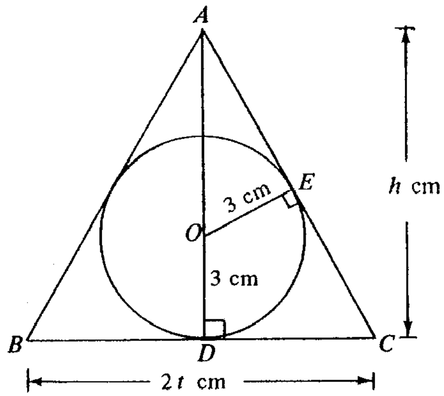

Figure 2(a)

ABC is a variable isosceles triangle with $AB = AC$ such that the radius of its inscribed circle is 3 cm. The height AD and the base BC of ΔABC are h cm and $2t$ cm respectively, where $h > 6$. (See Figure 2(a).) Let p cm be the perimeter of ΔABC.

(a) Show that $t^2 = \dfrac{9h}{h-6}$.

(3 marks)

(b) Show that $p = \dfrac{2h^{\frac{3}{2}}}{(h-6)^{\frac{1}{2}}}$.

(3 marks)

(c) Find

(i) the range of values of h for which $\dfrac{dp}{dh}$ is positive,

(ii) the minimum value of p.

(6 marks)

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

If you attempt Question 11, fill in the details in the first three boxes above and tie this sheet into your answer book.

(d) (i) In Figure 2(b), sketch the graph of p against h for $h > 6$.

(ii) Hence write down the range of values of p for which two different isosceles triangles whose inscribed circles are of radii 3 cm can have the same perimeter p cm.

(4 marks)

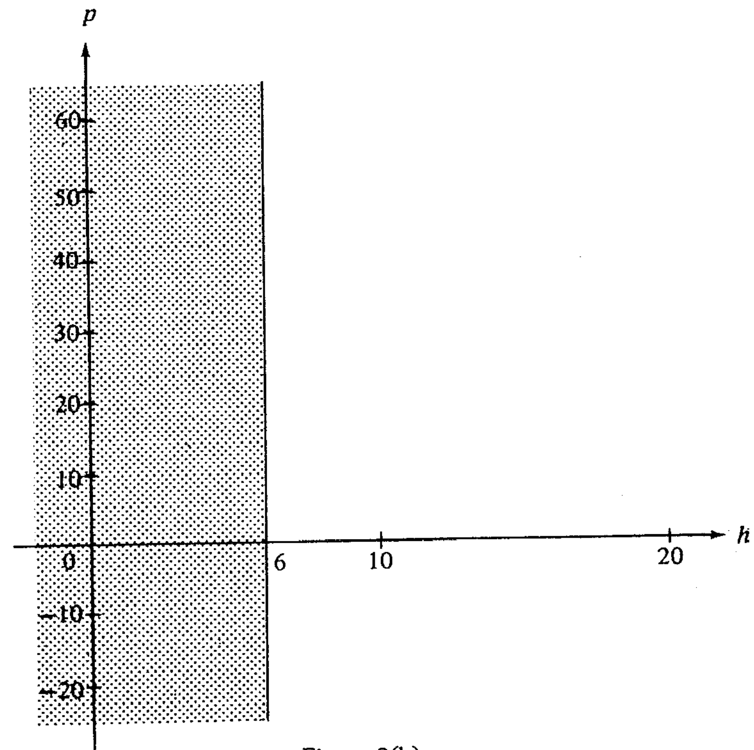

Figure 2(b)

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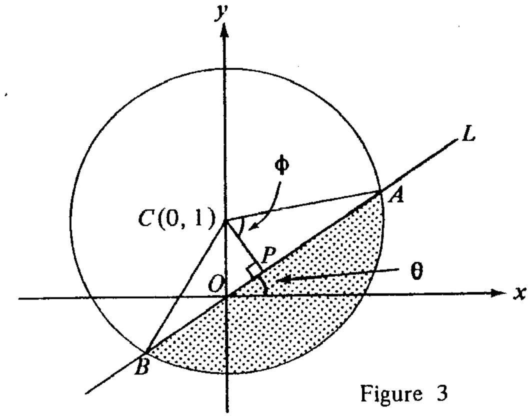

Figure 3

Figure 3 shows a circle of radius 2 centred at the point $C(0, 1)$. A variable straight line L with positive slope passes through the origin O and makes an angle θ with the positive x-axis. L intersects the circle at points A and B. Let S be the area of the shaded segment. P is the point on L such that CP is perpendicular to AB. Let $\angle PCA = \phi$.

(a) (i) Find the length of CP in terms of θ.
Hence show that $\cos\theta = 2\cos\phi$.

(ii) Show that $S = 4\phi - 2\sin 2\phi$.

(5 marks)

(b) (i) Find $\frac{d\phi}{d\theta}$ in terms of θ and ϕ.

(ii) Hence find $\frac{dS}{d\theta}$ in terms of θ.

(7 marks)

(c) L rotates about O in the clockwise direction such that θ decreases steadily at a rate of $\frac{1}{30}$ radian per second. Find the rate of change of S with respect to time when $\theta = \frac{\pi}{3}$.

(4 marks)

END OF PAPER

91-CE
A MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1991

ADDITIONAL MATHEMATICS PAPER II

11.15 am-1.15 pm (2 hours)
This paper must be answered in English

Answer ALL questions in Section A and any THREE questions in Section B.

All working must be clearly shown.

Unless otherwise specified in a question, numerical answers must be given in **exact value**.

Section A (42 marks)

Answer ALL questions in this section.

1. Given that $(1 + x + ax^2)^8 = 1 + 8x + k_1x^2 + k_2x^3 +$ terms involving higher powers of x.

   (a) Express k_1 and k_2 in terms of a.

   (b) If $k_1 = 4$, find the value of a.

   Hence find the value of k_2.

   (5 marks)

2. Evaluate $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$.

   (5 marks)

3. Find the general solution of

   $$\cos 4\theta + \cos 2\theta = \cos \theta.$$

   (5 marks)

4. A family of straight lines is given by the equation

   $$(2 - k)x + (1 + 2k)y - (4 + 3k) = 0,$$

   where k is any constant.

   Find equations of the two lines in the family whose distances from the origin are equal to 1.

   (5 marks)

5. The slope at any point (x, y) of a curve C is given by

$$\frac{dy}{dx} = 4 - 2x$$

and C passes through the point $(1, 0)$.

(a) Find an equation of C.

(b) Find the area of the finite region bounded by C and the x-axis.

(7 marks)

6.

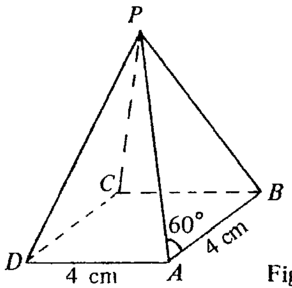

Figure 1

In Figure 1, $PABCD$ is a right pyramid with a square base of sides of length 4 cm. $\angle PAB = 60°$. Find, correct to the nearest 0.1 degree,

(a) the angle between the plane PAB and the base $ABCD$,

(b) the angle between the planes PAB and PAD.

(7 marks)

7. (a) Prove, by mathematical induction, that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all positive integers n.

(b) Using the formula in (a), find the sum

$$1 \times 2 + 2 \times 3 + \cdots + n(n + 1).$$

(8 marks)

Section B (48 marks)

Answer any **THREE** questions in this section.

Each question carries 16 marks.

8. (a) Given that $\tan x = k \tan y$.

(i) Show that $\sin(x + y) = (k + 1)\cos x \sin y$.

(ii) Hence show that

$$(k + 1)\sin(x - y) = (k - 1)\sin(x + y).$$

(6 marks)

(b) Given that

$$\tan(\theta + 10°) = k\tan(\theta - 20°)$$

has solutions in θ.

(i) Show that $\sin(2\theta - 10°) = \frac{k+1}{2(k-1)}$.

(ii) Hence find the range of possible values of k.

(7 marks)

(c) Find the general solution of

$$\tan(\theta + 10°) = -2\tan(\theta - 20°),$$

giving the answer correct to the nearest 0.1 degree.

(3 marks)

9. L is a straight line which passes through point $A(1, 2)$ and makes an angle θ with the positive x-axis. $P(x, y)$ is a point on L such that $AP = s$, as shown in Figure 2 (a).

(a) Write down the coordinates of P in terms of s and θ.

(2 marks)

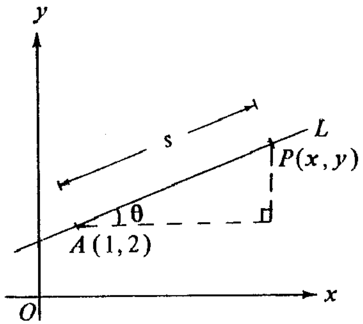

Figure 2(a)

The circle $C: x^2 + y^2 - 6x - 10y + 30 = 0$ cuts the line L at points H and K (see Figure 2 (b)). Let $AH = s_1$, $AK = s_2$.

(b) Show that s_1 and s_2 are the roots of the equation

$$s^2 - (4\cos\theta + 6\sin\theta)s + 9 = 0.$$

(3 marks)

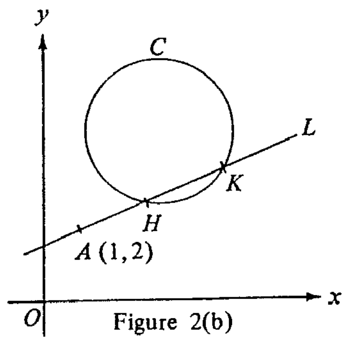

Figure 2(b)

(c) Using the result of (b), show that

$$HK^2 = 48\sin\theta\cos\theta - 20\cos^2\theta.$$

(4 marks)

(d) Using the result of (c), find equations of the two tangents from the point A to the circle C.

(7 marks)

10. Given a hyperbola $H: \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

and a straight line $L: 5x - 4y = 0$.

(a) A and B are two points on H. The tangents to H at A and B are parallel to L. Find the coordinates of A and B.

(7 marks)

(b) Let $y = \dfrac{5}{4}x + c$ be an equation of a straight line which cuts H at two points P and Q. Find the coordinates of the mid-point of PQ in terms of c.

(4 marks)

(c) A chord of H is parallel to L and M is its mid-point. Using the result of (b), find an equation of the locus of M. Sketch the locus.

(5 marks)

11.

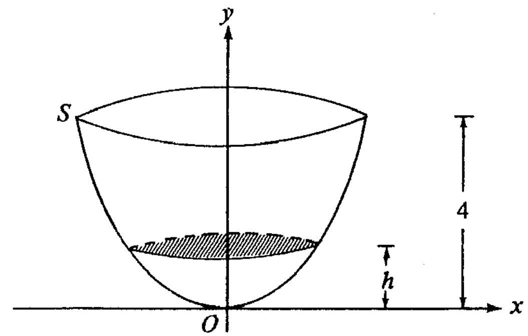

Figure 3(a)

An object S is in the shape of the solid of revolution of the region bounded by the curve $x^2 = 4y$ and the line $y = 4$ revolved about the y-axis, as shown in Figure 3 (a).

(a) Find the volume of S.

(3 marks)

(b) It is given that if S is cut by a plane parallel to its top surface at a distance h from O (see Figure 3 (a)), the mass of the part of S below the plane is given by

$$\int_0^h \pi(16y - 3y^2)\,dy, \quad \text{where } 0 < h \leq 4.$$

(i) Find the mass of S.

(ii) If the plane mentioned above cuts S into two parts of equal volumes, find

(1) the value of h,

(2) the ratio of the mass of the lower part to that of the upper part.

(9 marks)

(c)

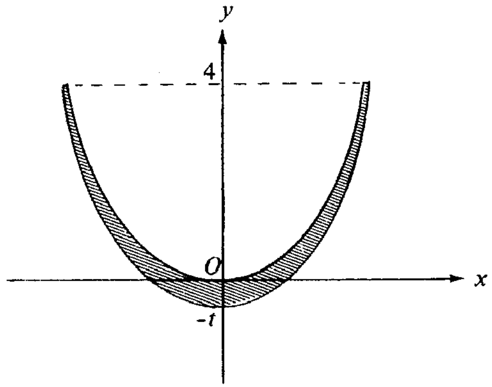

Figure 3(b)

In Figure 3(b), the shaded region is bounded by the curves $x^2 = 4y, x^2 = 4(y + t)$ and the line $y = 4$, where t is a small positive number. The solid of revolution of the shaded area revolved about the y-axis represents a layer of paint coated on the curved surface of S. Show that the volume of paint is approximately equal to $16\pi t$.

(4 marks)

12. Let m, n be positive integers.

(a) Given that $y = (1+x)^{m+1}(1-x)^n$. Find $\frac{dy}{dx}$.

Hence show that

$$(m+1)\int(1+x)^m(1-x)^n\,dx = (1+x)^{m+1}(1-x)^n + n\int(1+x)^{m+1}(1-x)^{n-1}\,dx.$$

(3 marks)

(b) Using the result of (a), show that

$$\int_{-1}^{1}(1+x)^m(1-x)^n dx = \frac{n}{m+1}\int_{-1}^{1}(1+x)^{m+1}(1-x)^{n-1}\,dx.$$

(3 marks)

(c) Without using a binomial expansion, evaluate

$$\int_{-1}^{1}(1+x)^8\,dx.$$

(2 marks)

(d) Using the substitution $x = \tan\theta$, show that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos^2 2\theta\,(1+\tan\theta)^4}{\cos^6\theta}\,d\theta = \int_{-1}^{1}(1+x)^6(1-x)^2\,dx.$$

Hence, using the results of (b) and (c), evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos^2 2\theta\,(1+\tan\theta)^4}{\cos^6\theta}\,d\theta.$$

(8 marks)

END OF PAPER

1991

Additional Mathematics I

1. $2 + 3i$

2. $-\frac{1}{(1+x)^2}$

3. $-3, -1, 2$

4. (a) $1 + 2\cos 2x,\ -4\sin 2x$

   (b) $\frac{\pi}{3} + \frac{\sqrt{3}}{2},\ \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

5. (a) $\frac{a + 3b}{4}$

   (b) (i) $\frac{k+1}{4k}(a + 3b)$

   (ii) $\frac{a + 2mb}{1 + m}$

   $\frac{5}{3}, \frac{3}{2}$

6. (a) $y = 2$

   $x = 1$

7. (a) $1 - 2k + k^2$

   (b) $x^2 - (2 - k)x + (1 - 2k + k^2) = 0$

   $0 \le k \le \frac{4}{3}$

8. (a) $(3 - x)\mathbf{i} - (y + 1)\mathbf{j}$

   $(x - 7)\mathbf{i} + (y - 1)\mathbf{j}$

   $(x - 10)\mathbf{i} + y\mathbf{j}$

   (b) (ii) (1) 4, 2

9. (a) $-2(x + 3)^2 - 5$

   (b) (i) $1, -\frac{3}{10}$

   (c) $\frac{3}{10}, -1$

10. (a) (i) $\cos\theta + i\sin\theta$, where $\theta = \pm\frac{2\pi}{3}$

    (ii) $\cos\theta + i\sin\theta$, where $\theta = \pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{8\pi}{9}$

11. (c) (i) $h > 9$

    (ii) $18\sqrt{3}$

    (d) (ii) $p > 18\sqrt{3}$

12. (a) (i) $2\cos\phi$

    (b) (i) $\frac{\sin\theta}{2\sin\phi}$

    (ii) $4\sin\theta\sqrt{1 - \frac{1}{4}\cos^2\theta}$

    (c) $\frac{-\sqrt{5}}{20}\,s^{-1}$

1991

Additional Mathematics II

1. (a) $8a + 28,\ 56a + 56$

   (b) $-3, -112$

2. $\frac{\pi}{2} + 1$

3. $2n\pi \pm \frac{\pi}{2},\ \frac{2n\pi}{3} \pm \frac{\pi}{9}$

4. $x = 1$

   $3x - 4y + 5 = 0$

5. (a) $y = -x^2 + 4x - 3$

   (b) $1\frac{1}{3}$

6. (a) 54.7°

   (b) 109.5°

7. (b) $\frac{1}{3}n(n + 1)(n + 2)$

8. (b) (ii) $k \ge 3$ or $k \le \frac{1}{3}$

   (c) $90n° + 5° + (-1)^n 4.8°$

9. (a) $(1 + s\cos\theta,\ 2 + s\sin\theta)$

   (d) $x = 1$

   $5x - 12y + 19 = 0$

10. (a) $(5, \frac{9}{4}),\ (-5, -\frac{9}{4})$

    (b) $(\frac{-5c}{4}, \frac{-9c}{16})$

    (c) $y = \frac{9x}{20}$ ($x > 5$ or $x < -5$)

11. (a) 32π

    (b) (i) 64π

    (ii) (1) $2\sqrt{2}$

    (2) $(2\sqrt{2} - 1) : 1$

12. (a) $(m + 1)(1 + x)^m(1 - x)^n - n(1 + x)^{m+1}(1 - x)^{n-1}$

    (c) $\frac{512}{9}$

    (d) $\frac{128}{63}$