HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1987

# 附加數學　試卷一
# ADDITIONAL MATHEMATICS　PAPER I

8.30 am–10.30 am　(2 hours)

This paper must be answered in English

Answer **ALL** questions in Section A and any **THREE** questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.



## SECTION A (39 marks)

**Answer ALL questions in this section.**

1. Let $f(x) = \operatorname{cosec}^2 3x$. Find $f'(\frac{\pi}{12})$.
(4 marks)

2. Let $x = y + \sin y$.

   Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $y$.
(5 marks)

3. For any complex number $z$, let $\bar{z}$, $|z|$ and $\operatorname{Re}(z)$ be its conjugate, modulus and real part respectively.

   Show that $z + \bar{z} = 2\operatorname{Re}(z)$ and $|z| \geqslant \operatorname{Re}(z)$.

   Hence, or otherwise, show that for any complex numbers $z_1$ and $z_2$,

   $$z_1 z_2 + \bar{z}_1 \bar{z}_2 \leqslant 2|z_1||z_2| .$$
(5 marks)

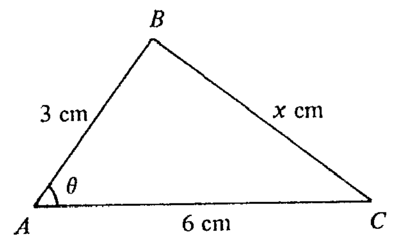
4. 


   **Figure 1**

   In Figure 1, $AB = 3$ cm, $AC = 6$ cm, $BC = x$ cm and $\angle A = \theta$.

   (a) Express $x^2$ in terms of $\theta$.

   (b) If $\theta$ increases at the rate of $\frac{1}{3}$ radian per second, find the rate of change of $x$ with respect to time when $\theta = \frac{\pi}{3}$.
(6 marks)

5. The equation $x^2 + 4x + p = 0$, where $p$ is a real constant, has distinct real roots $\alpha$ and $\beta$.

(a) Find the range of values of $p$.

(b) If $\alpha^2 + \beta^2 + \alpha^2\beta^2 + 3(\alpha + \beta) - 19 = 0$, find the value of $p$.

(6 marks)

6.

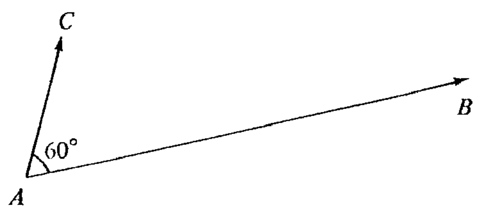



**Figure 2**

In Figure 2, $|\overrightarrow{AB}| = 3$, $|\overrightarrow{AC}| = 1$ and $\angle CAB = 60°$.

Find (a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$,

(b) $|\overrightarrow{AB} + 2\overrightarrow{AC}|$.

(6 marks)

7. Solve the inequality $(x + 2)|x - 2| < -5$.

(7 marks)



**SECTION B (60 marks)**

**Answer any THREE questions from this section.**
**Each question carries 20 marks.**

8.

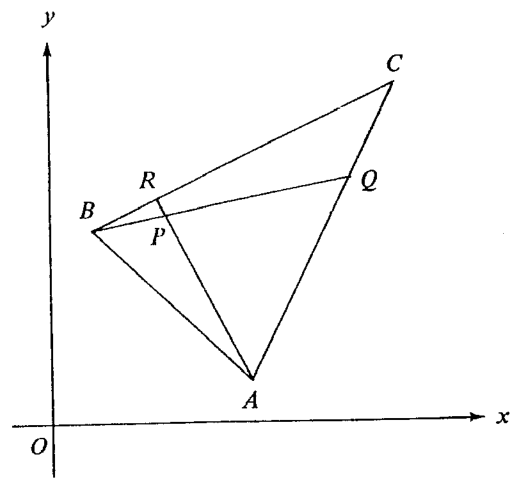



**Figure 3**

In Figure 3, $R$ is a point on $BC$ such that $BR : RC = m : 1$. $Q$ is a point on $AC$. $BQ$ intersects $AR$ at $P$. $\overrightarrow{OA} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{OB} = \mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OC} = 7\mathbf{i} + 7\mathbf{j}$ and $\overrightarrow{BQ} = 5\mathbf{i} + \mathbf{j}$.

(a) (i) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii) Express $\overrightarrow{AR}$ in terms of $m$, $\mathbf{i}$ and $\mathbf{j}$.

(4 marks)

(b) Suppose $AR$ is perpendicular to $BC$.

(i) Show that $m = \frac{1}{4}$.

(ii) Find $\angle QPR$.

(iii) If $\overrightarrow{BQ} = \lambda\overrightarrow{BA} + \mu\overrightarrow{BC}$, find the values of $\lambda$ and $\mu$.

(iv) If $AP : PR = n : 1$, express $\overrightarrow{BP}$ in terms of $n$, $\mathbf{i}$ and $\mathbf{j}$.

Hence find the value of $n$.

(16 marks)

9. (a)

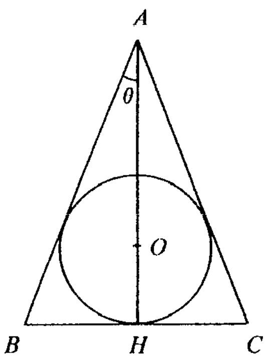



**Figure 4(a)**

Figure 4(a) shows a circle of centre $O$ and radius $a$ inscribed in an isosceles triangle $ABC$ with $AB = AC$. Let $\angle OAB = \theta$.

(i) Find, in terms of $a$ and $\theta$, the height $AH$ of $\triangle ABC$.

Hence show that the area of $\triangle ABC$ is

$$\frac{a^2(1+\sin\theta)^2}{\sin\theta\cos\theta}.$$

(ii) For what value of $\theta$ is the area of $\triangle ABC$ a minimum? (Testing for maximum/minimum is not required.) (10 marks)

(b)

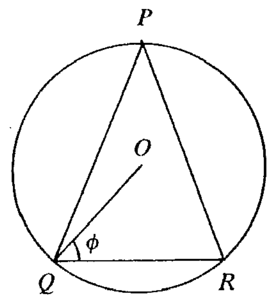



**Figure 4(b)**

Figure 4(b) shows a circle of centre $O$ and radius $b$ circumscribing an isosceles triangle $PQR$ with $PQ = PR$. Let $\angle OQR = \phi$.

(i) Show that the area of $\triangle PQR$ is

$$b^2\cos\phi(1+\sin\phi).$$

(ii) When $\triangle PQR$ is equilateral, show that its area is a maximum. (10 marks)



10. (a) Let $z = \cos\theta + i\sin\theta$, where $\theta$ is not a multiple of $\pi$.

If $z^2 - 2\bar{z} + \frac{1}{z}$ is real, find the two values of $z$. (9 marks)

(b) Let $z_1$ and $z_2$ be the two values of $z$ obtained in (a).

(i) Show that $z_1^2 = z_2$ and $z_2^2 = z_1$.

(ii) Find the values of $z_1^3$ and $z_2^3$.

(iii) Find the values of $z_1^k + z_2^k$ when

(1) $k = 3n$,
(2) $k = 3n+1$,
(3) $k = 3n+2$,

where $n$ is a positive integer.

(iv) For any positive integer $k$, show that

$$z_1^{2k} + z_2^{2k} = \begin{cases} 2 & \text{when } k \text{ is a multiple of } 3, \\ -1 & \text{when } k \text{ is not a multiple of } 3. \end{cases}$$

(11 marks)

11. It is given that the equation

$z^2 - 2z + k = 0$ ($k$ is real) ....................................(*)

has no real roots.

(a) Find the range of values of $k$. (2 marks)

(b) Find the quadratic equation whose roots are the cubes of the roots of (*) and show that the discriminant of this equation is $4(1-k)(4-k)^2$.

If this equation has real roots, deduce the value of $k$. (11 marks)

(c) Find, in terms of $k$, the squares of the roots of (*), expressing the answers in the form $x + iy$ where $x$ and $y$ are real.

As $k$ varies, find the equation of the locus of the points in the Argand plane representing the squares of the roots of (*). (7 marks)

12.

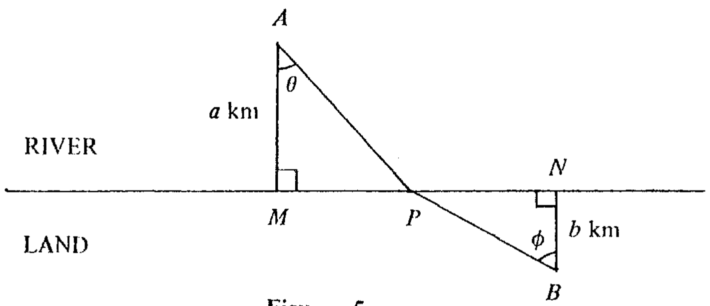



Figure 5

In Figure 5, $A$ is a fixed point in water $a$ km from a straight river bank. $B$ is a fixed point on land $b$ km from the river. $M$ and $N$ are the points on the bank nearest to $A$ and $B$ respectively. $P$ is a point between $M$ and $N$. Let $\angle MAP = \theta$ and $\angle NBP = \phi$. A man can swim at a speed of $u$ km/h and run at a speed of $v$ km/h, where $u < v$.

(a) The man swims from $A$ to $P$ and then runs to $B$.

(i) Express $MN$ in terms of $a$, $b$, $\theta$ and $\phi$.

Hence show that $\dfrac{d\phi}{d\theta} = -\dfrac{a\sec^2\theta}{b\sec^2\phi}$.

(ii) Let $t$ hours be the time taken to travel from $A$ to $B$ via $P$.

Show that $t = \dfrac{a}{u}\sec\theta + \dfrac{b}{v}\sec\phi$.

If $t$ is a minimum, show that $\dfrac{u}{v} = \dfrac{\sin\theta}{\sin\phi}$.

(Testing for maximum/minimum is not required.) (12 marks)

(b) Let $MN = h$ km. Suppose the man swims from $A$ to $P$ and then runs to $N$.

(i) Express the time taken in terms of $a$, $h$, $u$, $v$ and $\theta$.

(ii) Using the result in (b)(i), find $MP$ in terms of $a$, $u$ and $v$ when the time taken is a minimum.
(Testing for maximum/minimum is not required.) (5 marks)

(c) Suppose $C$ is a point in water $c$ km from $N$ and $CN \perp MN$. If the man swims from $A$ to $C$ via $P$ in the minimum time, find $MP : PN$. (3 marks)

**END OF PAPER**



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# 附加數學 試卷二
# ADDITIONAL MATHEMATICS PAPER II

11.15 am–1.15 pm (2 hours)
This paper must be answered in English

Answer **ALL** questions in Section A and any **THREE** questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.

## SECTION A (39 marks)

**Answer ALL questions in this section.**

1. If the coefficient of $x^2$ in the expansion of $(1 + x + x^2)^n$ is 21 and $n$ is a positive integer, find the value of $n$.

(5 marks)

2. Prove, by mathematical induction, that

$$\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1},$$

for all positive integers $n$.

(5 marks)

3. Find the equations of the two lines through $(1, 2)$, each making an angle of $\theta$ (where $\tan\theta = \frac{1}{2}$) with the line $3x - y = 0$.

(5 marks)

4. Using the substitution $x = \sin\theta$, evaluate $\int_0^{\frac{1}{2}} \frac{2x^2}{\sqrt{1-x^2}}\, dx$.

(6 marks)

5. The slope at any point $(x, y)$ of a curve is given by

$$\frac{dy}{dx} = (3x^2 - 2)(x^3 - 2x + 1)^{\frac{1}{3}}$$

If the curve passes through the origin, find the equation of the curve.

[Hint: Put $x^3 - 2x + 1 = u$.]

(6 marks)

6. Express $\sin 3\theta$ in terms of $\sin\theta$. Hence find the three roots of the equation $8x^3 - 6x + 1 = 0$ to 2 significant figures.

(6 marks)

7. Find the equations of the two tangents to the curve $x^2 - y^2 = 3$ which are parallel to the line $y = 2x$.

(6 marks)



## SECTION B (60 marks)

**Answer any THREE questions from this section.**
**Each question carries 20 marks.**

8. (a) Using the substitution $u = \tan x$, find

$$\int \tan^{n-2} x \sec^2 x\, dx,$$

where $n$ is an integer and $n \geqslant 2$.

(4 marks)

(b) (i) By writing $\tan^n x$ as $\tan^{n-2} x \tan^2 x$, show that

$$\int_0^{\frac{\pi}{4}} \tan^n x\, dx = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx,$$

where $n$ is an integer and $n \geqslant 2$.

(ii) Evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

(9 marks)

(c) Show that $\int_{-\frac{\pi}{4}}^{0} \tan^6 x\, dx = \int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

Hence evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^6 x\, dx$.

(7 marks)

9.

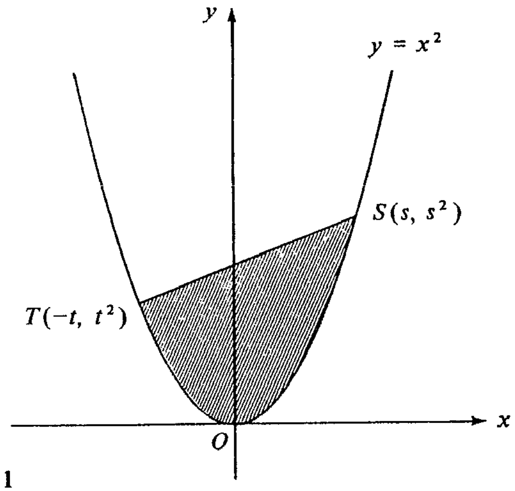


Figure 1

In Figure 1, $S(s, s^2)$ and $T(-t, t^2)$ are two points on the curve $y = x^2$, where $s$ and $t$ are positive real numbers.

(a) Find the area of the region bounded by the curve, the $x$-axis and the line $x = s$.

Hence, or otherwise, show that the area of the region bounded by the curve and the chord $ST$ (the shaded part in the figure) is $\frac{1}{6}(s + t)^3$. (7 marks)

(b) Suppose the chord $ST$ passes through $H(0, 1)$.

(i) Show that $t = \frac{1}{s}$.

(ii) Using the results in (a) and (b)(i), find the value of $s$ such that the area of the region bounded by the curve and the chord $ST$ is a minimum. (7 marks)

(c) If the region of minimum area in (b)(ii) is revolved about the $x$-axis, find the volume of the solid generated. (6 marks)



10.

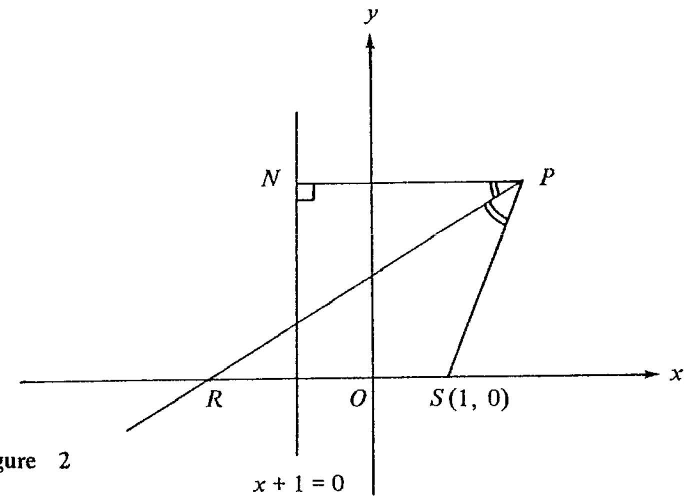


Figure 2

In Figure 2, $P(x, y)$ is a point equidistant from the point $S(1, 0)$ and the line $x + 1 = 0$.

(a) Show that the equation of the locus of $P$ is $y^2 = 4x$. (4 marks)

(b) Let the $y$-coordinate of $P$ be $2t$.

(i) Find the $x$-coordinate of $P$ in terms of $t$.

(ii) $N$ is the foot of the perpendicular from $P$ to the line $x + 1 = 0$. The bisector of $\angle SPN$ intersects the $x$-axis at $R$.

(1) Show that the equation of $PR$ is $x - ty + t^2 = 0$.

(2) Show that $PR$ touches $y^2 = 4x$ at $P$.

(3) Find the equation of the locus of the mid-point of $PR$. (16 marks)

11.

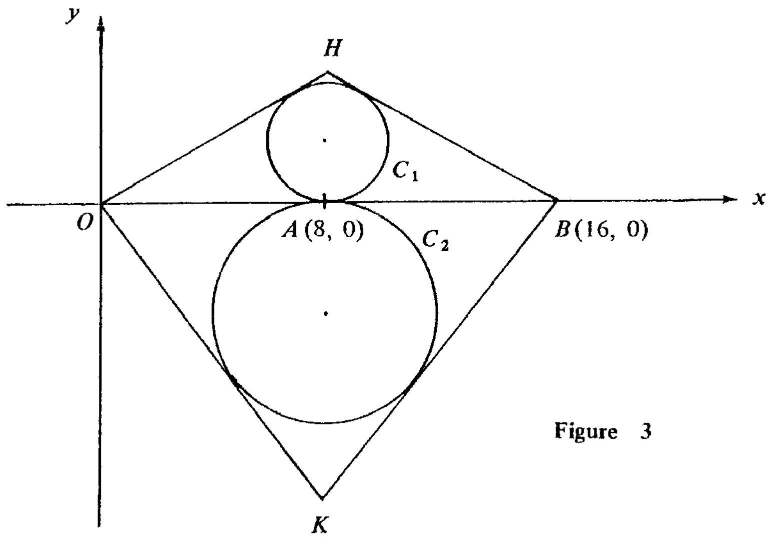


**Figure 3**

In Figure 3, $A$ and $B$ are the points (8, 0) and (16, 0) respectively. The equation of the circle $C_1$ is $x^2 + y^2 - 16x - 4y + 64 = 0$. $OH$ and $BH$ are tangents to $C_1$.

(a) (i) Show that $C_1$ touches the $x$-axis at $A$.

(ii) Find the equation of $OH$.

(iii) Find the equation of $BH$.

(12 marks)

(b) In the figure, the equation of $OK$ is $4x + 3y = 0$. The circle $C_2$ : $x^2 + y^2 - 16x + 2fy + c = 0$ is the inscribed circle of $\Delta OBK$ and touches the $x$-axis at $A$.

(i) Find the values of the constants $c$ and $f$.

(ii) Find area of $\Delta OBH$ : area of $\Delta OBK$.

(8 marks)



12. (a) (i) If $7 \sin \theta - 24 \cos \theta$ is expressed in the form $r \sin (\theta - A)$ where $r > 0$ and $0° < A < 90°$, find $r$ and $A$.

(ii) Let $y = 14 \sin \theta - 48 \cos \theta + 14$.

Using the result in (i), find the maximum and minimum values of $y$.

Find also the general values of $\theta$ at which $y$ attains its maximum.

(12 marks)

(b) $\alpha$ and $\beta$ are two acute angles satisfying the equation

$$6 \cos^2 \theta - 5 \cos \theta + 1 = 0.$$

Without solving the equation, show that

$$\cos \frac{\alpha + \beta}{2} + \cos \frac{\alpha - \beta}{2} = \sqrt{2}.$$

[Hint: $\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x)$.]

(8 marks)

**END OF PAPER**

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*Additional Mathematics Paper I*

1. $-12$

2. $\frac{dy}{dx} = \frac{1}{1 + \cos y}$

   $\frac{d^2y}{dx^2} = \frac{\sin y}{(1 + \cos y)^3}$

4. (a) $45 - 36\cos\theta$

   (b) $1\ (s^{-1})$

5. (a) $p < 4$

   (b) $-3$

6. (a) $\frac{3}{2}$

   (b) $\sqrt{19}$

7. $x < -3$

8. (a) (i) $\overrightarrow{AB} = -3\mathbf{i} + 3\mathbf{j}$

   $\overrightarrow{AC} = 3\mathbf{i} + 6\mathbf{j}$

   (ii) $\frac{1}{1+m}[(-3 + 3m)\mathbf{i} + (3 + 6m)\mathbf{j}]$

   (b) (ii) $105°$

   (iii) $\lambda = \frac{1}{3}$, $\mu = \frac{2}{3}$

   (iv) 10

9. (a) (i) $a + \frac{a}{\sin\theta}$

   (ii) $30°$

10. (a) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$,

    $\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$

    (b) (ii) $z_1{}^3 = 1$, $z_2{}^3 = 1$

    (iii) (1) 2

    (2) $-1$

    (3) $-1$

11. (a) $k > 1$

    (b) $z^2 + (6k - 8)z + k^3 = 0$

    $k = 4$

    (c) $(2 - k) \pm 2\sqrt{k-1}\,i$

    $y^2 = 4(1 - x)$

12. (a) (i) $a\tan\theta + b\tan\phi$

    (b) (i) $\frac{a\sec\theta}{u} + \frac{h - a\tan\theta}{v}$

    (ii) $\frac{au}{\sqrt{v^2 - u^2}}$

    (c) $a : c$

1987

*Additional Mathematics Paper II*

1. 6

3. $7x + y - 9 = 0$,

   $x - y + 1 = 0$

4. $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

5. $y = \frac{3}{4}(x^3 - 2x + 1)^{\frac{4}{3}} - \frac{3}{4}$

6. 0.17, 0.77, $-0.94$

7. $2x - y - 3 = 0$,

   $2x - y + 3 = 0$

8. (a) $\frac{\tan^{n-1} x}{n-1} + c$

   (b) (ii) $\frac{13}{15} - \frac{\pi}{4}$

   (c) $2(\frac{13}{15} - \frac{\pi}{4})$

9. (a) $\frac{s^3}{3}$

   (b) (ii) 1

   (c) $\frac{8\pi}{5}$

10. (b) (i) $t^2$

    (ii) (3) $x = 0$

11. (a) (ii) $8x - 15y = 0$

    (iii) $8x + 15y - 128 = 0$

    (b) (i) $c = 64$, $f = 4$

    (ii) $2 : 5$

12. (a) (i) $r = 25$, $A = 73.7°$

    (ii) Maximum value of $y = 64$

    Minimum value of $y = -36$

    $180n° + (-1)^n 90° + 73.7°$