HONG KONG CERTIFICATE OF EDUCATION EXAMINATION, 1986

ADDITIONAL MATHEMATICS PAPER I

Time allowed: Two Hours

SECTION A (39 marks)
Answer All questions in this section.

1. Find, from first principles, $\frac{\mathrm{d}}{\mathrm{d}x}(x^3)$.

(4 marks)

2. The quadratic equation

$$x^2 \log a + (x+1)\log b = 0,$$

where a and b are constants, has non-zero equal roots.
Find b in terms of a.

(5 marks)

3. The maximum value of the function $f(x) = 4k + 18x - kx^2$ (k is a positive constant) is 45. Find k.

(5 marks)

4. Find the equation of the tangent to the curve $x^2 + xy + y^2 = 7$ at the point $(2, 1)$.

(6 marks)

5. The angle between the two vectors $\mathbf{i} + \mathbf{j}$ and $(c+4)\mathbf{i} + (c-4)\mathbf{j}$ is θ, where $\cos\theta = -\frac{3}{5}$. Find the value of the constant c.

(6 marks)

6. On the same Argand diagram, sketch the locus of the point representing the complex number z in each of the following cases:

(a) $|z-2| = 1$;
(b) $|z-1| = |z-3|$.

Hence, or otherwise, find the complex numbers represented by the points of intersection of the two loci.

(6 marks)

7. Solve $x > \frac{3}{x} + 2$ for each of the following cases:

(a) $x > 0$;
(b) $x < 0$.

(7 marks)

SECTION B (60 marks)
Answer any THREE questions from this section. Each question carries 20 marks.

8. In Figure 1, $OACB$ is a trapezium with $OB \parallel AC$ and $AC = 2OB$. P and Q are points on OA and BC respectively such that $OP = \frac{1}{2}OA$ and $BQ = \frac{1}{3}BC$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

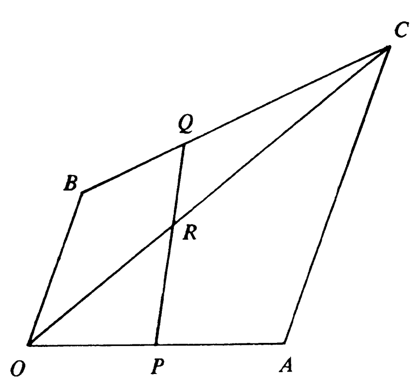

Figure 1

(a) Express $\overrightarrow{OC}$, $\overrightarrow{BC}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(5 marks)

(b) OC intersects PQ at the point R.
Let $PR : RQ = h : 1-h$.

(i) Express $\overrightarrow{OR}$ in terms of $\mathbf{a}, \mathbf{b}$ and h.

(ii) If $\overrightarrow{OR} = k\overrightarrow{OC}$, find h and k.

(9 marks)

(c) OB and PQ are produced to meet at T and $\overrightarrow{OT} = \lambda\mathbf{b}$.

(i) Express $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Express $\overrightarrow{PT}$ in terms of $\mathbf{a}, \mathbf{b}$ and λ.

(ii) Hence, or otherwise, find the value of λ.

(6 marks)

9. (a) Write down the general solution of the equation $\cos x = \dfrac{1}{\sqrt{2}}$.

(3 marks)

(b) Let m be a positive integer.

(i) If $z = r(\cos\theta + i\sin\theta)$, show that $z^m + \bar{z}^m = 2r^m\cos m\theta$.

(ii) By making use of (a) and (b)(i), or otherwise, find the values of m for which

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^m + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^m = \sqrt{2}.$$

(9 marks)

(c) (i) Let p be a positive integer. Find the values of p for which

$$(1+i)^p - (1-i)^p = 0.$$

(ii) By making use of (c)(i), or otherwise, find the value of

$$\frac{(1+i)^{4k+1}}{(1-i)^{4k-1}},$$

where k is a positive integer.

(8 marks)

10. The graph of the function $f(x) = x^3 + hx^2 + kx + 2$ (h and k are constants) has 2 distinct turning points and intersects the line $y = 2$ at the point $(0, 2)$ only.

(a) Show that $3k < h^2 < 4k$.

(8 marks)

(b) It is also known that the graph of $f(x)$ passes through $(-2, 0)$.

(i) Express k in terms of h.

(ii) If h is an integer, use the results in (a) and (b)(i) to show that $h = 4$ or 5.

(iii) For $h = 4$, find the maximum and minimum points of the graph of $f(x)$ and sketch this graph.

(12 marks)

11. Figure 2 shows two rods OP and PR in the xy-plane. The rods, each 10 cm long, are hinged at P. The end O is fixed while the end R can move along the positive x-axis. $OL = 20$ cm, $OR = s$ cm and $\angle POR = \theta$, where $0 \leq \theta \leq \dfrac{\pi}{2}$.

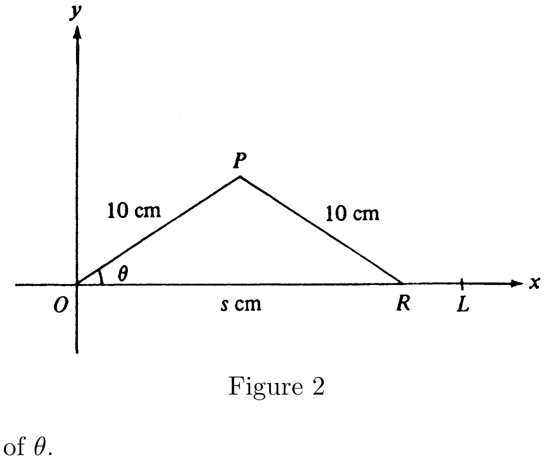

Figure 2

(a) Express s in terms of θ.
If R moves from the point O to the point L at a speed of 10 cm/s, find the rate of change of θ with respect to time when $s = 10$.

(5 marks)

(b) Find the equation of the locus of the mid-point of PR and sketch this locus.

(5 marks)

(c) A square of side ℓ cm is inscribed in $\triangle OPR$ such that one side of the square lies on OR. Show that

$$\ell = \frac{20\sin\theta\cos\theta}{\sin\theta + 2\cos\theta}.$$

Hence find θ when the area of the square is a maximum.

(10 marks)

12. Figure 3 shows a rectangular picture of area A cm^2 mounted on a rectangular piece of cardboard of area 3600 cm^2 with sides of length x cm and y cm. The top, bottom and side margins are 12 cm, 13 cm and 8 cm wide respectively.

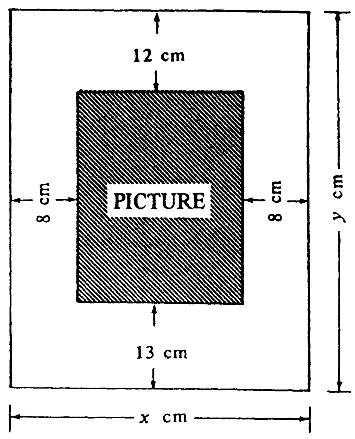

Figure 3

(a) Find A in terms of x.

(2 marks)

(b) Show that the largest value of A is 1600.

(5 marks)

(c) (i) Find the range of values of x for which A decreases as x increases.
(ii) If $x \geq 50$, find the largest value of A.

(6 marks)

(d) If $\frac{4}{9} \leq \frac{x}{y} \leq \frac{9}{16}$, find the range of values of x and the largest value of A.

(7 marks)

END OF PAPER

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION, 1986

ADDITIONAL MATHEMATICS PAPER II

Time allowed: Two Hours

SECTION A (39 marks)
Answer All questions in this section.

1. Prove, by mathematical induction, that for any positive integer n,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

(5 marks)

2. In the expansion of $(x^2+2)^n$ in descending powers of x, where n is a positive integer, the coefficient of the third term is 40. Find the value of n and the coefficient of x^4.

(5 marks)

3. If θ is an obtuse angle and the equation in x

$$3x^2-(4\cos\theta)x+2\sin\theta=0$$

has equal roots, find the value of θ.

(5 marks)

4. Using the identity $\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$, find the general solution of $\sin 2\theta+\sin 4\theta=\cos\theta$.

(6 marks)

5. $A(3,6)$, $B(-1,-2)$ and $C(5,-3)$ are three points. $P(s,t)$ is a point on the line AB.
   (a) Find t in terms of s.
   (b) If the area of $\triangle APC$ is $\frac{13}{2}$, find the two values of s.

(6 marks)

6. A straight line through $C(3,2)$ with slope m cuts the curve $y=(x-2)^2$ at the points A and B. If C is the mid-point of AB, find the value of m.

(6 marks)

7. Let $y=\frac{\tan^3\theta}{3}-\tan\theta$.
   Find $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ in terms of $\tan\theta$.
   Hence, or otherwise, find $\int\tan^4\theta\,\mathrm{d}\theta$.

(6 marks)

SECTION B (60 marks)
Answer any THREE questions from this section. Each question carries 20 marks.

8. (a) Show that $\int_0^a f(x)\,\mathrm{d}x=\int_0^a f(a-x)\,\mathrm{d}x$.

(4 marks)

   (b) Using the result in (a), or otherwise, evaluate the following integrals:
      (i) $\int_0^\pi \cos^{2n+1}x\,\mathrm{d}x$, where n is a positive integer,
      (ii) $\int_0^\pi x\sin^2 x\,\mathrm{d}x$,
      (iii) $\int_0^{\frac{\pi}{2}}\frac{\sin x\,\mathrm{d}x}{\sin x+\cos x}$.

(16 marks)

9. A family of straight lines is given by the equation

$$(3k+2)x-(2k-1)y+(k-11)=0,$$

where k is any constant.

   (a) L_1 is the line $x-2y+4=0$.

(i) There are two lines in the family each making an angle of 45° with L_1. Find the equations of these lines.

(ii) Find the equation of the line L in the family which is parallel to L_1.
The line L_1 and another line L_2 are equidistant from L. Find the equation of L_2.

(11 marks)

(b) For what value of k does the line in the family form a triangle of minimum area with the two positive coordinate axes?

(7 marks)

(c) The straight lines in the family pass through a fixed point Q. Write down the equation of the line which passes through Q but which does not belong to the family.

(2 marks)

10. The circles $C_1 : x^2 + y^2 - 4x + 2y + 1 = 0$ and $C_2 : x^2 + y^2 - 10x - 4y + 19 = 0$ have a common chord AB.

(a) (i) Find the equation of the line AB.

(ii) Find the equation of the circle with AB as a chord such that the area of the circle is a minimum.

(9 marks)

(b) The circle C_1 and another circle C_3 are concentric. If AB is a tangent to C_3, find the equation of C_3.

(4 marks)

(c) $P(x, y)$ is a variable point such that

$$\frac{\text{distance from } P \text{ to the center of } C_1}{\text{distance from } P \text{ to the center of } C_2} = \frac{1}{k} \qquad (k > 0).$$

Find the equation of the locus of P.

(i) When $k = 2$, write down the equation of the locus of P and name the locus.

(ii) For what value of k is the locus of P a straight line?

(7 marks)

11. (a) (i) Using the substitution $x = 2\sin\theta$, evaluate

$$\int_1^2 \sqrt{4 - x^2}\, dx.$$

(ii) Express $3 + 2x - x^2$ in the form $a^2 - (x - b)^2$ where a and b are constants.
Using the substitution $x - b = a\sin\theta$, evaluate

$$\int_0^1 \sqrt{3 + 2x - x^2}\, dx.$$

(11 marks)

(b) In Figure 1, the shaded region is bounded by the two circles $C_1 : x^2 + y^2 = 4$, $C_2 : (x - 1)^2 + (y - \sqrt{3})^2 = 4$ and the parabola $S : y^2 = 3x$.

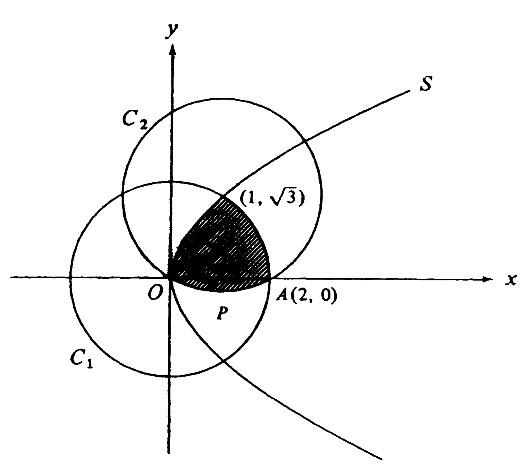

Figure 1

(i) $P(x, y)$ is a point on the minor arc OA of C_2. Express y in terms of x.

(ii) Find the area of the shaded region.

(9 marks)

12. (a) (i) Express $\sin 108°$ in terms of $\sin 36°$.

Using this result and the relation $\sin 108° = \sin 72°$, show that $\cos 36° = \dfrac{1+\sqrt{5}}{4}$.

(ii) Find $\cos 72°$ in surd form.

(7 marks)

(b) In Figure 2, O is the center of the circle APB of radius 1 unit. $AKON, AHB$ and KHP are straight lines. PK and BN are both perpendicular to AN. $\angle AOH = 60°$. Let $\angle OAH = \phi$.

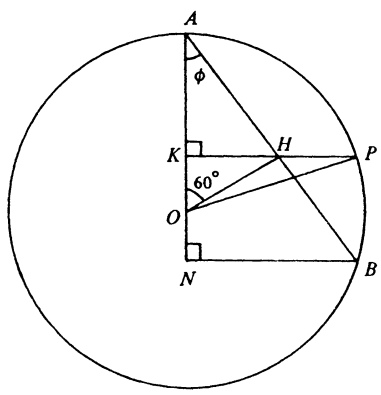

Figure 2

(i) By considering $\triangle AOH$, express OH in terms of $\tan\phi$.

Hence show that $\cos \angle POK = \dfrac{\tan\phi}{\sqrt{3}+\tan\phi}$.

(ii) It is given that $ON = \frac{1}{4}$.

(1) Find BN and hence the value of $\tan\phi$. Give your answers in surd form.

(2) Find the value of $\cos \angle POK$ in surd form.
Hence find $\angle POK$ without using calculators.

(13 marks)

END OF PAPER