HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1985

附加數學　試卷一
ADDITIONAL MATHEMATICS　PAPER I

8.30 am–10.30 am　(2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.

SECTION A (39 marks)

Answer ALL questions in this section.

1. Let $f(x) = x\sqrt{1 - x^2}$. Find the value of $f'(\frac{1}{2})$.

(5 marks)

2. Express $1 - i$ in polar form.

Hence find the cube roots of $1 - i$ (give your answers in polar form).

(6 marks)

3. Solve the inequality $x^2 - ax - 4 \leqslant 0$, where a is real. If among the possible values of x satisfying the above inequality, the greatest is 4 , find the least.

(6 marks)

4. In Figure 1 , $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i} - 3\mathbf{j}$. C is a point on AB such that $\frac{AC}{CB} = r$.

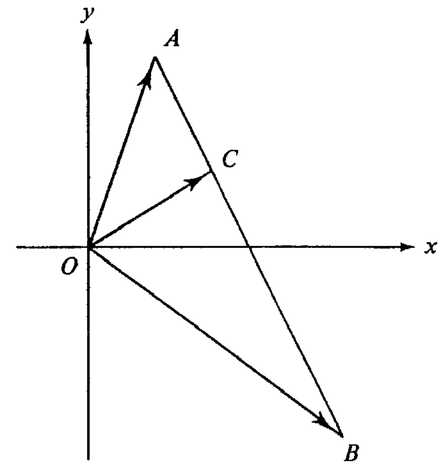

Figure 1

(a) Express $\overrightarrow{OC}$ in terms of r .

(b) Find the value of r if OC is perpendicular to AB . Hence find the coordinates of C .

(6 marks)

5.

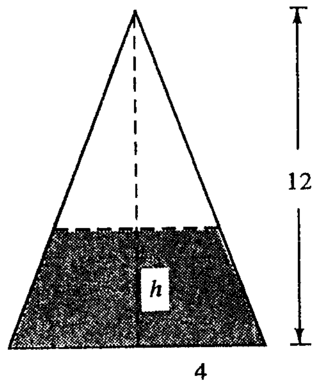

Figure 2

Figure 2 shows a vessel in the shape of a right circular cone with base radius 4 cm and height 12 cm. Water is poured into the vessel through the apex. Find the volume of the water in the vessel when the depth of the water is h centimetres. If water is poured into the vessel at a rate of π cm^3/s, how fast is the water level rising when the depth of the water is 6 cm ?

(8 marks)

6. Find the two real values of p for which the equation

$$\log_{10} \left| x^2 + 2px \right| = 0$$

has a double root.

(8 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

7. (a) In the equation $ax^2 + bx + c = 0$, a, b and c are complex numbers and $a \neq 0$.

(i) By the method of completing the square, factorize the expression $ax^2 + bx + c$.

(ii) Show that if a, b and c are real numbers such that $b^2 - 4ac < 0$, then the given equation has imaginary roots.

(iii) Show that if $a = 3i$, $b = -2$ and $c = 5i$, then $b^2 - 4ac > 0$, but the equation still has imaginary roots.

(9 marks)

(b) The equation

$$x^2 - 2\lambda x + (2\lambda^2 - 2\lambda\mu + \mu^2) = 0$$

has real roots. If λ and μ are real, find the relation between them.

(4 marks)

(c) The coefficients of the following equations are real :

$$x^2 + ax + b = 0 \quad \ldots\ldots\ldots\ldots (1)$$
$$x^2 + cx + d = 0 \quad \ldots\ldots\ldots\ldots (2)$$
$$2x^2 + (a + c)x + (b + d) = 0 \quad \ldots\ldots (3).$$

Prove that if the roots of (1) and (2) are imaginary, so are the roots of (3).

(7 marks)

8. In Figure 3, ΔOBA is right-angled at B. OB is produced to C such that $OB = BC$. CD is drawn in the direction of OA such that $CD = kOA$. P is a point on AD such that $CP \parallel BA$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{DP} = \lambda\overrightarrow{DA}$.

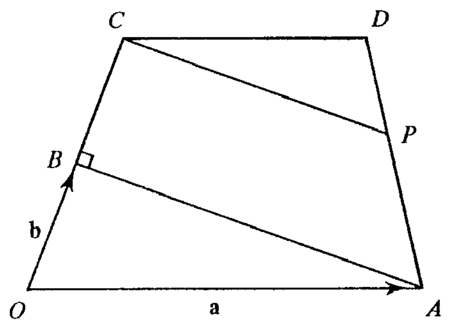

Figure 3

(a) (i) Express $\overrightarrow{OD}$ and $\overrightarrow{DA}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k.

(ii) Find $\overrightarrow{BA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and express $\overrightarrow{CP}$ in terms of $\mathbf{a}$, $\mathbf{b}$, λ and k.

Hence find λ in terms of k.

(9 marks)

(b) (i) Show that $\mathbf{a} \cdot \mathbf{b} = OB^2$.

(ii) If $OB = \frac{1}{4}OA$, show that $\overrightarrow{OD} \cdot \overrightarrow{DA} = (-16k^2 + 12k - 2)OB^2$.

Hence find the values of k and λ if $OD \perp DA$.

(11 marks)

9. (a) In Figure 4, $P(a, b)$ is a point in the first quadrant. A variable line segment QR passes through P with the end Q on the x-axis and R on the y-axis. Let $\angle RQO = \theta$ and $QR = s$.

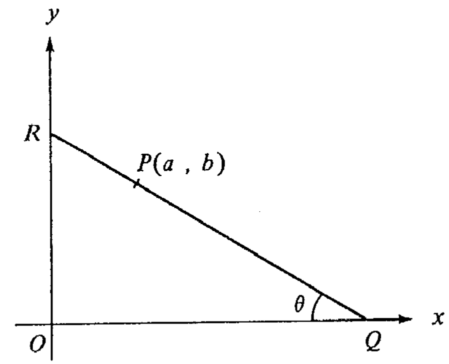

Figure 4

(i) Express s in terms of a, b and θ.

(ii) Show that s will be least when $\tan\theta = \sqrt[3]{\frac{b}{a}}$. (11 marks)

(b) Figure 5 shows two corridors meeting at right angles. The width of one corridor is 0.8 m and that of the other is 2.7 m. A pipe is to be moved from one corridor into the other.

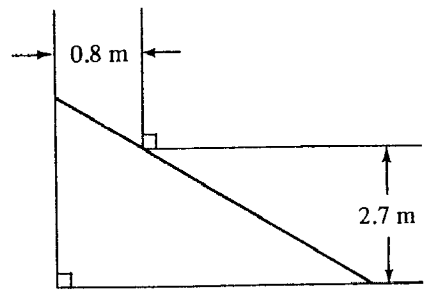

Figure 5

(i) If the pipe is to lie completely on the horizontal floor when it is being moved round the corner, what is the greatest possible length of the pipe?

(ii) If the height of the ceiling of each corridor is 3 m, find the length of the longest pipe that can be carried round the corner. (9 marks)

10. Let z be a complex number not equal to -1 and $w = \frac{z-1}{z+1}$.

(a) Let $w = p + qi$, where p and q are real.

Show that $p = \frac{1}{2}(w + \bar{w})$

and $q = \frac{1}{2i}(w - \bar{w})$.

Hence show that $p = \frac{z\bar{z} - 1}{z\bar{z} + z + \bar{z} + 1}$

and $q = \frac{i(\bar{z} - z)}{z\bar{z} + z + \bar{z} + 1}$.

(7 marks)

(b) In each of the following cases, find the locus of z and interpret the result geometrically :

(i) w is real,

(ii) w is purely imaginary,

(iii) $|w| = 1$. [Hint : You may use $|w|^2 = w\bar{w}$.]

(13 marks)

11. In Figure 6 , BD is an advertisement painted on a vertical wall BDC of a building. $BD = 20$ m , $DC = 10$ m . An observer at A , x metres from the wall, finds the angle subtended by the advertisement at his eye to be θ .

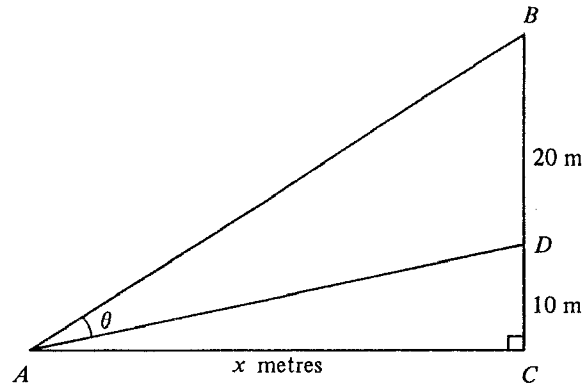

Figure 6

(a) Show that $\tan\theta = \frac{20x}{x^2 + 300}$.

(3 marks)

(b) By differentiating both sides of the result in (a) with respect to x , show that $\frac{d\theta}{dx} = \frac{20(300 - x^2)}{x^4 + 1000x^2 + 90\,000}$.

Hence find the value of x for which θ is a maximum.

(6 marks)

(c) Find the value of $\frac{d\theta}{dx}$ at $x = 50$, correct to 4 decimal places.

Hence estimate the increase in the distance between the observer and the wall if the angle subtended is to be decreased by 1° from that observed at $x = 50$ (your answer should be correct to the nearest $\frac{1}{10}$ m) .

(6 marks)

(d) Sketch the graph of θ against x for $x \geqslant 0$.

(5 marks)

END OF PAPER

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HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1985

附加數學　試卷二
ADDITIONAL MATHEMATICS PAPER II

11.15 am–1.15 pm (2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.

SECTION A (39 marks)

Answer ALL questions in this section.

1. $(ax + \frac{1}{x^2})^n$ is expanded in descending powers of x, where n is a positive integer and $a > 0$. If the fourth term of the expansion is independent of x and is equal to $\frac{21}{2}$, find the values of n and a. (5 marks)

2. Let $T_n = \frac{n(n+2)}{(n+1)^2}$, where n is a positive integer. Prove by mathematical induction that

$$T_1 \times T_2 \times \ldots \times T_n = \frac{n+2}{2(n+1)}$$

for all n.

(5 marks)

3. Using the substitution $u = 25 - x^2$, evaluate $\int_3^4 \frac{x}{\sqrt{25 - x^2}}\,dx$.
(5 marks)

4. In Figure 1, the straight line L cuts the x-axis at the point $(-1, 0)$ and the curve $y = 4 - x^2$ at the point $(1, 3)$. Find the area of the shaded part.

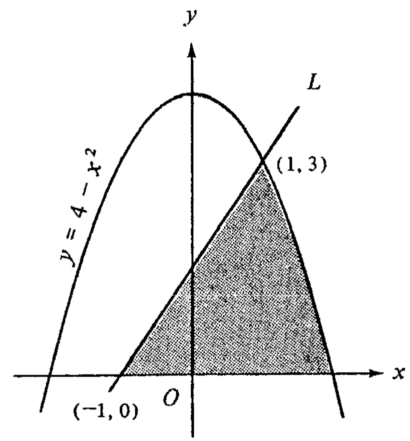


Figure 1

(5 marks)

5. The line $y = x$ and the circle $x^2 + y^2 - 2y = 0$ intersect at the points A and B. Write down the equation of the family of circles passing through A and B.

Hence find the equations of the two circles passing through these two points and with radius $\sqrt{5}$ (6 marks)

6. Find the equations of the two tangents drawn from the point $(-1, 0)$ to the parabola $y^2 = 4x$. (6 marks)

7. In triangle ABC, $\angle A$ and $\angle B$ are acute, $\sin A = \frac{5}{13}$ and $\sin B = \frac{3}{5}$.

(a) Show that $\sin C = \sin(A + B)$ and hence find the value of $\sin C$ without using calculators.

(b) If the perimeter of the triangle is 12 cm, find the length of the longest side. (7 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

8. (a) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$.

(4 marks)

(b) By using the substitution $t = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt .$$

(4 marks)

(c) Show that $\int_{-\frac{\pi}{2}}^{0} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt$

and $\int_{-\frac{\pi}{2}}^{0} \sin^3 t \cos^4 t \, dt = -\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$.

(6 marks)

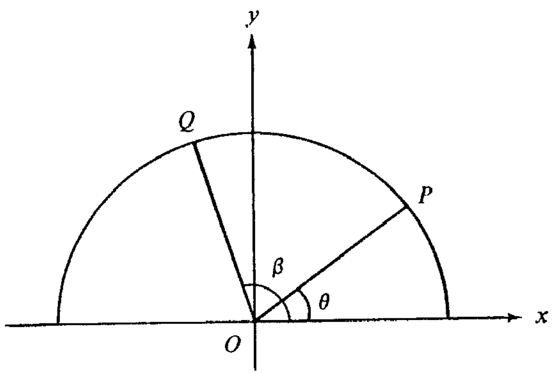
(d) Using the above results, or otherwise, evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} \sin^3 2t \, (\sin t + \cos t) \, dt .$$

(6 marks)

9. In Figure 2, P and Q are two points on the semi-circle $y = \sqrt{1 - x^2}$. OP and OQ make angles θ and β respectively with the positive x-axis, where $\theta \leqslant \beta$.

Figure 2



(a) The region bounded by OP, OQ and arc PQ is revolved about the x-axis. Show that the volume of the solid generated is $\frac{2\pi}{3}(\cos\theta - \cos\beta)$.

(10 marks)

(b) If P and Q move along the semi-circle such that $\beta = 2\theta$, find the maximum volume of the solid.

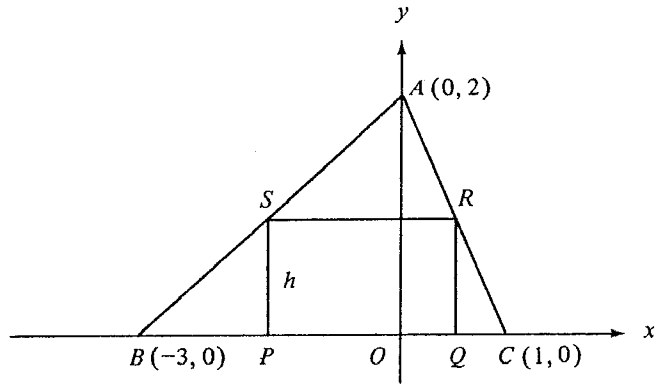
(5 marks)

(c) If P and Q move along the semi-circle such that $\beta - \theta = \frac{\pi}{3}$, find the maximum volume of the solid.

(5 marks)

10. $A(0, 2)$, $B(-3, 0)$ and $C(1, 0)$ are the vertices of a triangle. $PQRS$ is a variable rectangle inscribed in the triangle with PQ on the x-axis, R on AC and S on AB, as shown in Figure 3. Let the length of PS be h.

Figure 3



(a) Find the coordinates of S and R in terms of h.

(5 marks)

(b) Let A_1 be the area of $PQRS$ when it is a square, A_2 be the maximum possible area of rectangle $PQRS$, and A_3 be the area of ΔABC. Find the ratios $A_1 : A_2 : A_3$.

(8 marks)

(c) The centre of $PQRS$ is the point $M(x, y)$. Express x and y in terms of h.

Hence find the equation of the locus of M.

Show the locus on a diagram.

(7 marks)

11. The line $y = 2x + c$ cuts the ellipse $x^2 + \frac{y^2}{16} = 1$ at the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(a) (i) Show that $PQ = \sqrt{5}\,|x_1 - x_2|$.

(ii) Show that x_1 and x_2 are the roots of the equation

$$20x^2 + 4cx + (c^2 - 16) = 0 .$$

(iii) Determine the two values of c such that the length of the chord PQ is $2\sqrt{2}$.

(11 marks)

(b) Let the two chords determined in (a)(iii) be $P'Q'$ and $P''Q''$. $P'Q'Q''P''$ is a parallelogram as shown in Figure 4.

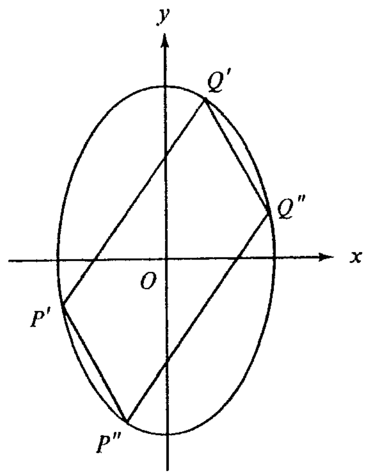


Figure 4

(i) By finding the distance between the chords $P'Q'$ and $P''Q''$, or otherwise, calculate the area of the parallelogram.

(ii) By finding the relation between the coordinates of P'' and the coordinates of Q', or otherwise, calculate the length of the side $P'P''$.

(9 marks)

12. In Figure 5, $ABCDE$ is a regular pentagon of side 1 cm inscribed in a circle with centre O. H is the foot of the perpendicular drawn from C to the diagonal BE.

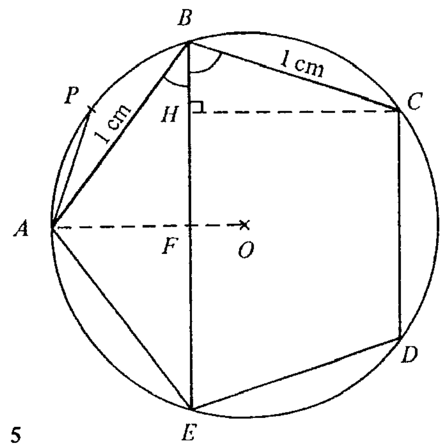


Figure 5

(a) Find $\angle ABE$ and $\angle CBE$.

By expressing the length of BE in two different forms, prove that $\cos 36° - \cos 72° = \frac{1}{2}$.

Hence find the value of $\cos 36°$ in surd form.

(9 marks)

(b) Show that the radius of the circle is $\frac{2}{\sqrt{10 - 2\sqrt{5}}}$ cm.

(5 marks)

(c) Let AP be one side of a regular decagon (10-sided polygon) inscribed in the same circle. Find $\angle PAO$, and hence show that

$$AP = \frac{2}{\sqrt{10 + 2\sqrt{5}}} \text{ cm}.$$

(6 marks)

END OF PAPER

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Additional Mathematics I

1. $\frac{1}{\sqrt{3}}$

2. $\sqrt{2}\,(\cos -\frac{\pi}{4} + i\sin -\frac{\pi}{4})$

   $\sqrt[6]{2}\left(\cos\frac{(8k-1)\pi}{12} + i\sin\frac{(8k-1)\pi}{12}\right)$,

   $k = 0 , 1 , 2$

3. $\frac{a - \sqrt{a^2+16}}{2} \leqslant x \leqslant \frac{a + \sqrt{a^2+16}}{2}$

   -1

4. (a) $\frac{1}{1+r}\,[(1+4r)\mathbf{i} + (3-3r)\mathbf{j}]$

   (b) $r = \frac{1}{2}$

   $C = (2 , 1)$

5. $\frac{\pi}{27}(432h - 36h^2 + h^3)$ cm^3

   $\frac{1}{4}$ cm/s

6. ± 1

7. (a) (i) $a\left(x + \frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b+\sqrt{b^2-4ac}}{2a}\right)$

   (iii) The roots are $\frac{5}{3i}$ and $\frac{-1}{i}$

   (b) $\lambda = \mu$

8. (a) (i) $\overrightarrow{OD} = 2\mathbf{b} + k\mathbf{a}$

   $\overrightarrow{DA} = (1-k)\mathbf{a} - 2\mathbf{b}$

   (ii) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

   $\overrightarrow{CP} = [k + \lambda(1-k)]\,\mathbf{a} - 2\lambda b$

   $\lambda = \frac{k}{1+k}$

   (b) (ii) $k = \frac{1}{4}$ or $\frac{1}{2}$

   $\lambda = \frac{1}{5}$ or $\frac{1}{3}$

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Additional Mathematics I

9. (a) (i) $s = a\sec\theta + b\,\mathrm{cosec}\,\theta \quad (0 < \theta < \frac{\pi}{2})$

   (b) (i) 4.69 m

   (ii) 5.57 m

10. (b) (i) $z - \bar{z} = 0$
    The locus of z is the real axis, excluding $z = -1$.

    (ii) $z\bar{z} - 1 = 0$
    The locus of z is the circle, centre O, radius 1, excluding the points $z = \pm 1$.

    (iii) $z + \bar{z} = 0$
    The locus of z is the imaginary axis.

11. (b) $\sqrt{300}$

    (c) -0.0050

    3.5

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Additional Mathematics II

1. $n = 9$

   $a = \frac{1}{\sqrt{2}}$

3. 1

4. $4\frac{2}{3}$

5. $x^2 + y^2 - 2y + k(x - y) = 0$

   $x^2 + y^2 + 2x - 4y = 0$

   $x^2 + y^2 - 4x + 2y = 0$

6. $x - y + 1 = 0$

   $x + y + 1 = 0$

7. (a) $\frac{56}{65}$

   (b) 5.6 cm

8. (a) $\frac{2}{35}$

   (d) $\frac{4}{35}$

9. (b) $\frac{3}{4}\pi$

   (c) $\frac{2\pi}{3}$

10. (a) $S = (\frac{3h}{2} - 3 , h)$

    $R = (1 - \frac{h}{2} , h)$

    (b) 8 : 9 : 18

    (c) $x = \frac{1}{2}(h - 2)$

    $y = \frac{h}{2}$

    $x - y = -1$

11. (a) (iii) $\pm\sqrt{10}$

    (b) (i) 8 (sq. units)

    (ii) $\sqrt{26}$

12. (a) $\angle ABE = 36^\circ$

    $\angle CBE = 72^\circ$

    $\frac{1+\sqrt{5}}{4}$

    (c) $\angle PAO = 72^\circ$