香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八四年香港中學會考

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION, 1984

ADD. MATHEMATICS PAPER I

MARKING SCHEME

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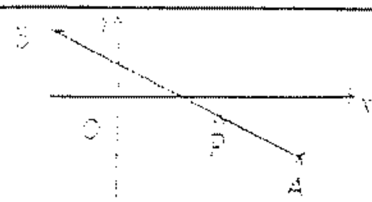
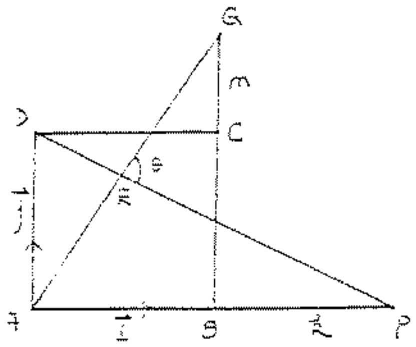
SOLUTIONS	MARKS	REMARKS
1. (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$	1	
$= (-\vec{i} - \vec{j}) - (3\vec{i} - 2\vec{j})$		
$= -4\vec{i} + 3\vec{j}$	1A	
The unit vector $= \frac{-4\vec{i} + 3\vec{j}}{\sqrt{(-4)^2+3^2}}$	1M	Method to find unit vector
$= -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$	1A	
		Alternatively:
(b) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$	1	$\overrightarrow{OP} = m\vec{b} + (1-m)\vec{a}$ 2A
$= \overrightarrow{OA} + m\,\overrightarrow{AB}$	1M	$= m(-\vec{i}+\vec{j}) + (1-m)(3\vec{i}-2\vec{j})$
$= (3\vec{i} - 2\vec{j}) + (-4m\vec{i} + 3m\vec{j})$		$= (3-4m)\vec{i} + (3m-2)\vec{j}$ 1A
$= (3 - 4m)\vec{i} + (3m - 2)\vec{j}$	1A	
	7	
2. $S = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$		
$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}$	1A+1A	
$\therefore 8\pi r \frac{dr}{dt} = 8$	1M	Equate $\frac{dS}{dt} = 8$
At $S = 36\pi$, $r = 3$	1A	
$\frac{dr}{dt} = \frac{8}{8\pi r} = \frac{1}{3\pi}$	1A	
$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$		
$= 4\pi r^2 \cdot \frac{dr}{dt}$	1A	
At $S = 36\pi$, $\frac{dV}{dt} = 36\pi \cdot \frac{1}{3\pi}$	1M	Sub for r
$= 12$		
The volume is increasing at a rate of 12 cm^3/s	1A	
	8	
Alternatively		
$S = 4\pi r^2$		
$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot (\frac{S}{4\pi})^{\frac{3}{2}}$	1M+2A	1M for attempt to eliminate r
$\frac{dV}{dt} = \frac{dV}{dS} \cdot \frac{dS}{dt}$	1	
$= \frac{4}{3}\pi \cdot (\frac{1}{4\pi})^{\frac{3}{2}} \cdot \frac{3}{2} S^{\frac{1}{2}} \cdot \frac{dS}{dt}$	1A	
$= \frac{4}{3}\pi (\frac{1}{4\pi})^{\frac{3}{2}} \cdot \frac{3}{2} (36\pi)^{\frac{1}{2}} \cdot 8$	1M+1A	1M for sub $S = 36\pi$, $\frac{dS}{dt} = 8$
$= 12$	1A	
The volume is increasing at a rate of 12 cm^3/s	1A	

SOLUTIONS	MARKS	REMARKS
3. (a) $z = (1 - 2i)^5$		
$= 1 + 5(-2i) + 10(-2i)^2 + 10(-2i)^3$ $+ 5(-2i)^4 + (-2i)^5$	1A	
$= (1 - 40 + 80) + (-10 + 80 - 32)i$		
$= 41 + 38i$	1A	
(b) $\frac{1}{z} = \frac{1}{41 + 38i}$		
$= \frac{1}{(41)^2 + (38)^2}(41 - 38i)$	1M	Rationalisation
$Re(\frac{1}{z}) = \frac{41}{41^2 + 38^2}$	1A	
$= \frac{41}{3125}$ (= 0.013)		
$Re(z + \frac{1}{z}) = 41 + \frac{41}{3125}$	1M	$Re(z + \frac{1}{z}) = Re(z) + Re(\frac{1}{z})$
$\doteq 41.013$		This step may be omitted if correct answer is directly given.
= 41 (correct to the nearest integer)	1A	
	5	

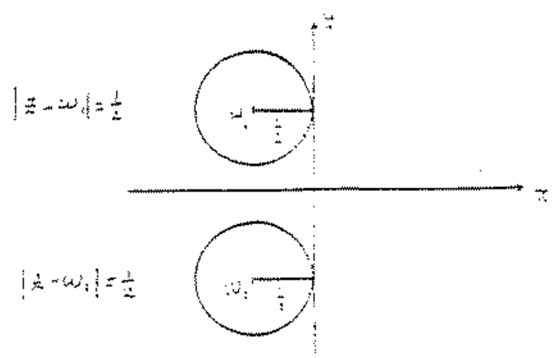
SOLUTIONS	MARKS	REMARKS
4. $\lvert 2\lvert x\rvert - 1\rvert \leq 2$		
$-2 \leq 2\lvert x\rvert - 1 \leq 2$	1A+1A	
$-1 \leq 2\lvert x\rvert \leq 3$		
$-\frac{1}{2} \leq \lvert x\rvert \leq \frac{3}{2}$ (or $\lvert x\rvert \leq \frac{3}{2}$)	1A	
$\therefore -\frac{3}{2} \leq x \leq \frac{3}{2}$	1A+1A	-1 if strict inequality ' < ' given in any line
	5	
Alternatively		
(i) Let $2\lvert x\rvert - 1 \geq 0$, then $\lvert x\rvert \geq \frac{1}{2}$		
i.e. $x \geq \frac{1}{2}$ or $x \leq -\frac{1}{2}$		
$\lvert 2\lvert x\rvert - 1\rvert \leq 2 \implies 2\lvert x\rvert - 1 \leq 2$		
$\implies \lvert x\rvert \leq \frac{3}{2}$		
$\implies -\frac{3}{2} \leq x \leq \frac{3}{2}$	1A	
Combining with the assumption,		
$\frac{1}{2} \leq x \leq \frac{3}{2}$ or $-\frac{3}{2} \leq x \leq -\frac{1}{2}$.	1A	
(ii) Let $2\lvert x\rvert - 1 < 0$, then $\lvert x\rvert < \frac{1}{2}$		
i.e. $-\frac{1}{2} < x < \frac{1}{2}$.		
$\lvert 2\lvert x\rvert - 1\rvert \leq 2 \implies 1 - 2\lvert x\rvert \leq 2$		
$\implies -\frac{1}{2} \leq \lvert x\rvert$		
This is true for all x .	1A	
$\therefore -\frac{1}{2} < x < \frac{1}{2}$.	1A	
Combining (i) and (ii),		
$-\frac{3}{2} \leq x \leq \frac{3}{2}$	1A	

SOLUTIONS	MARKS	REMARKS
5. (a) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$	1A	Alternatively:
$= 1^2 + 4(m^2 - m + 1)$ (sub $\alpha+\beta$, $\alpha\beta$) $\longrightarrow$	1M+1M	$D = 4 + 4(m^2 - m + 1)$
$= 4(m^2 - m + 2)$		$= 4(m^2 - m + 2)$..........1A
$= 4\left[(m-\frac{1}{2})^2 + \frac{7}{4}\right]$ completing square $\longrightarrow$	1M+1A	> 0 because the
> 0 ($\forall m \in R$)		discriminant of $4(m^2-m+2) = 0$
		is negative and 1M
		coeff. of m^2 is positive..1M
		$\therefore \alpha, \beta$ are real & distinct...1M
		Hence $(\alpha-\beta)^2 > 0$......1A
(b) Since $(\alpha-\beta)^2$ is real, $\lvert\alpha-\beta\rvert = \sqrt{(\alpha-\beta)^2}$		
Minimum value of $\lvert\alpha-\beta\rvert$		
is $\sqrt{7}$.	1M+1A	1M for $(m-\frac{1}{2})^2 = 0$
	7	
		Alternatively:
6. Let $AP = x$. Since $AB = AC$,		Let $PD = x$.
$AD \perp BC$ and $\angle BAD = \angle CAD = \theta$	1	$AD \perp BC$, $\angle BAD = \angle CAD = \theta$
$PE = PF = x\sin\theta$.	1A	$PE = PF = (h-x)\sin\theta$
Product of distances $p = x^2\sin^2\theta\,(h-x)$	1M	$p = x(h-x)^2\sin^2\theta$
$\frac{dp}{dx} = \sin^2\theta\,(2xh - 3x^2)$		$\frac{dp}{dx} = (h^2-4hx+3x^2)\sin^2\theta$
$= x\sin^2\theta\,(2h-3x)$		$= (h-3x)(h-x)\sin^2\theta = 0$
$\frac{dp}{dx} = 0 \iff x = 0$ or $\frac{2}{3}h$	1M+1A	$\frac{dp}{dx} = 0 \Rightarrow x = h$ or $\frac{1}{3}h$
At $x = \frac{2}{3}h$, $\frac{dp}{dx}$ changes sign from +ve to -ve.	1A	Max. at $x = \frac{1}{3}h$
$\therefore p$ is a maximum at $x = \frac{2}{3}h$.		(working necessary)
	6	

SOLUTIONS	MARKS	REMARKS
7. (a) $\overrightarrow{BQ} = (1+m)\vec{j}$		
$\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{BQ}$		
$= \vec{i} + (1+m)\vec{j}$		
$\overrightarrow{AP} = (1+k)\vec{i}$	2A+1A	
$\overrightarrow{DP} = \overrightarrow{DA} + \overrightarrow{AP}$		
$= (1+k)\vec{i} - \vec{j}$		
$\overrightarrow{AQ} \cdot \overrightarrow{DP} = (1+k) - (1+m)$	1M	
$= k - m$	1A	
But $\overrightarrow{AQ} \cdot \overrightarrow{DP} = \lvert\overrightarrow{AQ}\rvert\lvert\overrightarrow{DP}\rvert \cos\theta$	1	
$= \sqrt{m^2+2m+2}\,\sqrt{k^2+2k+2}\,\cos\theta$	1M	
$k - m = \sqrt{(m^2+2m+2)(k^2+2k+2)}\,\cos\theta$		
$\Rightarrow \cos\theta = \dfrac{k-m}{\sqrt{(m^2+2m+2)(k^2+2k+2)}}$	1A	
	8	
(b) (i) $\overrightarrow{AE} = \frac{1}{5}(\overrightarrow{AP} + 4\overrightarrow{AD})$	1	
$= \frac{1+k}{5}\vec{i} + \frac{4}{5}\vec{j}$	1A	
(ii) $\overrightarrow{AE} = r\,\overrightarrow{AQ}$	1	
$= r(\vec{i} + (1+m)\vec{j})$		
$= r\vec{i} + r(1+m)\vec{j}$	1A	
(iii) If $\theta = 90°$, $\cos\theta = 0$	1A	
$\therefore k = m$	1A	
Equating $\overrightarrow{AE}$ in (b)(i) and (ii),	1M	
$\frac{1+k}{5} = r$, $\frac{4}{5} = r(1+m)$.	1A	
From 1st equation $1 + k = 5r$		Attempt to solve............1M
From 2nd equation $\frac{4}{5r} = 1 + m = 1 + k$		$r = \frac{2}{5}$ 1A
$5r = \frac{4}{5r}$	1M+1A + 1A+1A	$k = 1$ 1A
$r^2 = \frac{4}{25}$		$m = 1$ 1A
$r = \frac{2}{5}$ (−ve root rejected)		−1 for not rejecting negative roots
$m = k = 1$		
	12	



SOLUTIONS	MARKS	REMARKS
8. (a) $f(\frac{1}{2}) = 5(\frac{1}{2})^2 + b(\frac{1}{2}) + c$	1	
$\therefore \frac{5}{4} + \frac{b}{2} + c < 0$	1A	
Consider $5x^2 + bx + c = 0$ Discriminant $= b^2 - 20c$	1A	
$> b^2 + 20(\frac{5}{4} + \frac{b}{2})$	1M	(sub. c)
$= b^2 + 10b + 25$		
$= (b + 5)^2$	1M	
Thus the discriminant is always positive		
$\therefore f(x) = 0$ has two distinct real roots.	1A	
	6	
(b) (i) $f(x) = 5x^2 + bx + c$		The omission of the factor 5 will not be penalised again.
$= 5(x - \alpha)(x - \beta)$	2A	
Since $f(\frac{1}{2}) < 0$		
$(\frac{1}{2} - \alpha)(\frac{1}{2} - \beta) < 0$	1M	
$\therefore$ either $\beta < \frac{1}{2} < \alpha$ or $\alpha < \frac{1}{2} < \beta$	1A	Accept " , "
Since $\alpha < \beta$,		
$\alpha < \frac{1}{2} < \beta$	1A	Explanation necessary
Further $\alpha\beta = \frac{c}{5}$	1	
$\therefore \alpha\beta > 0$	1A	
and $\alpha > 0$ as $\beta > 0$	1A	
$\therefore 0 < \alpha < \frac{1}{2} < \beta$		
	8	
(b) (ii) $\lvert\alpha - \frac{1}{2}\rvert = \lvert\beta - \frac{1}{2}\rvert$		
$\Rightarrow \frac{1}{2} - \alpha = \beta - \frac{1}{2}$	1A	
$\Rightarrow \alpha + \beta = 1$	1A	
$\therefore b = -5(\alpha + \beta)$	1M	
$= -5$	1A	
$\frac{5}{4} + \frac{b}{2} + c < 0 \Rightarrow c < \frac{5}{4}$	1M	
$\therefore 0 < c < \frac{5}{4}$	1A	
	6	

SOLUTION	MARKS	REMARKS
9. (a) (i) $w^3 = 1$	1	
$\Rightarrow w^3 - 1 = 0$		
$\Rightarrow (w-1)(w^2+w+1) = 0$	1A	
$\Rightarrow w^2+w+1 = 0$ since $w \neq 1$	1A	must mention $w \neq 1$
(ii) $(w^2)^{3k+1} + w^{3k+1} + 1$		
$= (w^3)^{2k} \cdot w^2 + (w^3)^k w + 1$	1M	Factorise powers of w^3
$= w^2 + w + 1 = 0$	1A	
$(w^2)^{3k+2} + w^{3k+2} + 1$		
$= (w^3)^{2k} \cdot w^4 + w^{3k} w^2 + 1$		
$= w + w^2 + 1$		
$= w^2 + w + 1 = 0$	1A	
	6	
(b) $\lvert 1 - w\bar{z} \rvert^2 = (1 - w\bar{z})\overline{(1 - w\bar{z})}$	1	
$= (1 - w\bar{z})(1 - \bar{w}z)$		
$= 1 - w\bar{z} - \bar{w}z + w\bar{w}z\bar{z}$		
$= 1 - w\bar{z} - \bar{w}z + z\bar{z}$	1A	
$\lvert z - w \rvert^2 = (z - w)\overline{(z - w)}$	1	
$= (z - w)(\bar{z} - \bar{w})$		
$= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}$		
$= z\bar{z} - z\bar{w} - \bar{z}w + 1$	1A	
$\therefore \lvert 1 - w\bar{z} \rvert = \lvert z - w \rvert$	1A	
	5	
(c) $\lvert 1 - w\bar{z} \rvert = c$		
$\Leftrightarrow \lvert z - w \rvert = c$	2	
which is a circle with	1A	
centre w and radius c	1A	
Let $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad (= \text{cis}\ \frac{2}{3}\pi)$	1A	
$w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \quad (= \text{cis}\ -\frac{2}{3}\pi)$	1A	
	3	Deduct 1 mark for ommision of each of the following features: (i) 2 circles in appropriate quadrants (ii) radius $= \frac{1}{2}$ (iii) touching y-axis (iv) not cutting x-axis note (iii) $\Rightarrow$ (ii)
	9	

(b) Alternative 1:

Let $z = x + iy$, $w = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$w\bar{z} = (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)(x - iy)$

$= (-\frac{x}{2} + \frac{\sqrt{3}y}{2}) + (\frac{y}{2} + \frac{\sqrt{3}x}{2}) i$ 1A

$|1 - w\bar{z}|^2 = (1 + \frac{x}{2} - \frac{\sqrt{3}y}{2})^2 + (\frac{y}{2} + \frac{\sqrt{3}x}{2})^2$

$= 1 + \frac{x^2}{4} + \frac{3y^2}{4} + x - \sqrt{3}y - \frac{\sqrt{3}xy}{2} + \frac{y^2}{4} + \frac{3x^2}{4} + \frac{\sqrt{3}xy}{2}$

$= x^2 + y^2 + x - \sqrt{3}y + 1$ 1A

$|z - w|^2 = (x^2 + x + \frac{1}{4}) + (y^2 - \sqrt{3}y + \frac{3}{4})$

$= x^2 + y^2 + x - \sqrt{3}y + 1$ 1A

$\therefore |1 - w\bar{z}| = |z - w|$ 1A

Let $w = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$,

$|1 - w\bar{z}|^2$ = etc.

Answer 1A

Alternative 2:

Let $z = r \operatorname{cis} \theta$

$w = \operatorname{cis} \phi$

$|1 - w\bar{z}|^2 = 1 + |w\bar{z}|^2 - 2|w\bar{z}| \cos(\phi - \theta)$ 1A

$= 1 + |z|^2 - 2|z| \cos(\phi - \theta)$ 1A

$|w - z|^2 = |w|^2 + |z|^2 - 2|w||\bar{z}| \cos(\phi - \theta)$ 1A

$= 1 + |z|^2 - 2|z| \cos(\phi - \theta)$ 1A

$\therefore |1 - w\bar{z}| = |w - z|$ 1A

Alternative 3:

$|1 - w\bar{z}| = |w(\frac{1}{w} - \bar{z})|$ 1A

$= |\frac{1}{w} - \bar{z}|$ ($\because |w| = 1$) 1A

$= |\bar{w} - \bar{z}|$ 1A

$= |w - z|$ 1A

$= |z - w|$ 1A

Alternative 4:

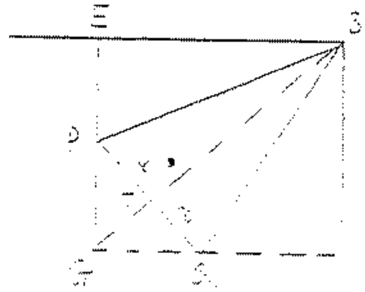
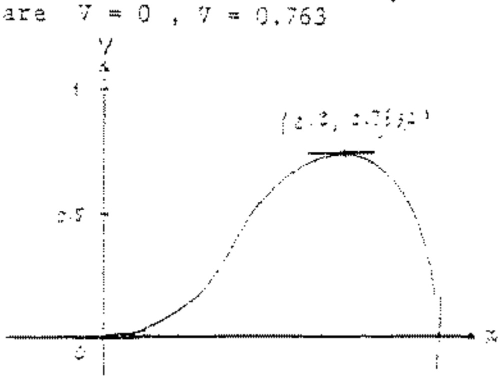
$|1 - w\bar{z}| = |w\bar{w} - w\bar{z}|$ 1A

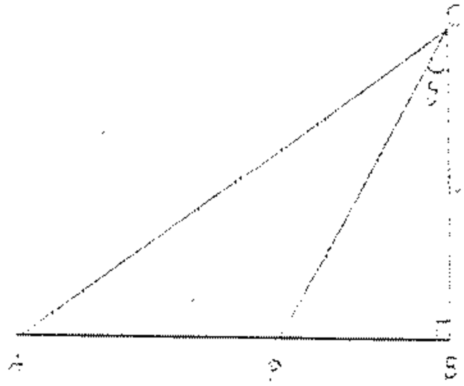
$= |w||\bar{w} - \bar{z}|$ 1M

$= |\bar{w} - \bar{z}|$ 1M

$= |w - z|$ 1A

$= |z - w|$ 1A

SOLUTIONS	MARKS	REMARKS
10. (a) Let G be the centre of the two		Alternatively:
squares and BG bisects PQ at T		$PG = \frac{1}{2}(4x^2 + 4x^2)^{\frac{1}{2}}$
$BG = \frac{1}{2}\sqrt{AB^2 + BC^2}$	1	$= x\sqrt{2}$ 1A
$= \frac{1}{2}\sqrt{8 + 8}$		$PE = \sqrt{2}(1 - x)$.........1A
$= 2$	1A	$BP = \sqrt{BE^2 + PE^2}$
$\therefore BT = 2 - x$	1A	$= \sqrt{4 - 4x + 2x^2}$ 1A
The height of the pyramid		Height $= \sqrt{BP^2 - PG^2}$....2
$= \sqrt{BT^2 - TG^2}$	2	$= \sqrt{4 - 4x + 2x^2 - 2x^2}$..1A
$= \sqrt{(2 - x)^2 - x^2}$	1A	$= 2\sqrt{1 - x}$
$= 2\sqrt{1 - x}$ metres		
Volume $= \frac{1}{3}$ base area X height		
$V = \frac{1}{3} \times (2x)^2 \times 2\sqrt{1 - x}$	1M	
$= \frac{8}{3}x^2\sqrt{1 - x}$	1A	
	8	
(b) $\frac{dV}{dx} = \frac{8}{3}\left(2x\sqrt{1 - x} - \frac{x^2}{2\sqrt{1 - x}}\right)$	1M	
$= \frac{4x(4 - 5x)}{3\sqrt{1 - x}}$	1A	Follow if constant factor of V incorrect.
$\frac{dV}{dx} = 0$ iff $x = 0$ or $\frac{4}{5}$	1A	
$\therefore$ the stationary points are		
$(0, 0)$, $\left(\frac{4}{5}, \frac{128}{75\sqrt{5}}\right)$	1A	Accept (0.8, 0.763)
At $x = 1$, $V = 0$,		
and the slope is infinite	1A	
$\therefore$ Equation of tangent at $x = 1$ is $x - 1 = 0$	1A	
Equations of tangent at stationary points are $V = 0$, $V = 0.763$	1A+1A	
 Graph of V(x)	4	1 mark for slope at (0, 0), 1 mark for slope at (1, 0), 1 mark for range, 1 mark for maximum with coordinates labelled.
	12	

SOLUTIONS	MARKS	REMARKS
11. (a) $CP = h \sec\theta \quad \left(\frac{h}{\cos\theta}\right)$	1A	
$PB = h\tan\theta$		
$AP = 30 - h\tan\theta$	1A	
$N = 2h\sec\theta + (30 - h\tan\theta)$ $N = 2CP + AP$	1M+1A	
	4	
(b) If $h = 50$, $N = 100\sec\theta + 30 - 50\tan\theta$		
$\frac{dN}{d\theta} = 2h\sec\theta\tan\theta - h\sec^2\theta$	1A	
$= 100\sec\theta\tan\theta - 50\sec^2\theta$		
$\frac{dN}{d\theta} = 0 \Rightarrow 50\sec\theta(2\tan\theta - \sec\theta) = 0$	1M	
$\Rightarrow 2\tan\theta - \sec\theta = 0$		
$\Rightarrow \sin\theta = \frac{1}{2}$	1A	
$\Rightarrow \theta = \frac{\pi}{6}$ (30°)	1A	
$\frac{d^2N}{d\theta^2} = 100(\sec^3\theta + \sec\theta\tan^2\theta - \sec^2\theta\tan\theta)$	1M	Compulsory, however, follow if omitted or wrong
$= \frac{200}{\sqrt{3}} > 0$ at $\theta = \frac{\pi}{6}$	1A	
∴ N is least at $\theta = \frac{\pi}{6}$. The least transportation cost from C to A is		
$\$\left(100 \times \frac{2}{\sqrt{3}} + 30 - 50 \times \frac{1}{\sqrt{3}}\right)$		
$= \$\,30\left(\frac{3+\sqrt{3}}{\sqrt{3}}\right) = \$\,30(\sqrt{3}+1)$	1A	
	7	
(c) (i) As $0 \leq \theta \leq \angle ACB$	1M	$AP = 30 - h\tan\theta \geq 0$ 1M
$\tan\theta \leq \tan\angle ACB$	1A	$\tan\theta \leq \frac{30}{h}$ 1A
$= \frac{30}{h}$		
For $h > 30\sqrt{3}$, $\tan\theta < \frac{1}{\sqrt{3}}$	1A	
$\therefore \theta < \frac{\pi}{6}$	1A	
Hence $\frac{dN}{d\theta} = h\sec\theta(2\tan\theta - \sec\theta)$		
$= \frac{h}{\cos^2\theta}(2\sin\theta - 1)$	1A	
< 0 as $\sin\theta < \frac{1}{2}$	1A	
(ii) If $h = 200$		
then $h > 30\sqrt{3}$	1A	
$\therefore \frac{dN}{d\theta} < 0$	1	
∴ N decreases as θ increases. Goods should be transported directly from C to A by truck for minimum cost.	1A	
	9	

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ADD. MATHEMATICS (PAPER II)

MARKING SCHEME

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SOLUTION	MARKS	REMARKS
1. $(x^2 + \frac{a}{x})^8 =$		
$= x^{16} + 8x^{14}\frac{a}{x} + 28x^{12}\frac{a^2}{x^2} + 56x^{10}\frac{a^3}{x^3} + \ldots$		~~1 for B_7~~
$= x^{16} + 8ax^{13} + \underline{28a^2x^{10}} + \underline{56a^3x^7} + \ldots$	3A	~~1 for B_{10}~~ 2A+1A ~~1 for the rest~~
$56a^3 = 4 \times 28a^2$	1M	<u>Alternatively:</u>
$\therefore a = 2$ (as $a \neq 0$)	1A	The general term = $_8C_r\, a^r\, x^{16-3r}$ $16-3r = 7 \Rightarrow r = 3$ $\therefore B_7 = {}_8C_3\, a^3 = \underline{56\, a^3}$ $16-3r = 10 \Rightarrow r = 2$ } 2A+1A $\therefore B_{10} = {}_8C_2\, a^2 = \underline{28a^2}$ etc.
	5	
2. If $n = 1$, $4n^3 - n = 3$, which is divisible by 3.	1A	[illegible] marks if this part is [illegible]
Assume that 3 divides $4k^3-k$ for some positive integer k.	1A	
Let $4k^3 - k = 3m$, where m is an integer.		
$4(k+1)^3 - (k+1) = 4(k^3+3k^2+3k+1) - (k+1)$		
$= (4k^3-k) + 3(4k^2+4k+1)$	1M+1A	1M for using assumption
$= 3m + 3(4k^2 + 4k + 1)$	1A	
$= 3(m + 4k^2 + 4k + 1)$, which is divisible by 3		
can be omitted <u>By induction</u>, 3 divides $4n^3-n$ for all positive integers n.	1A	
	6	

SOLUTION	MARKS	REMARKS
3. $y = \int (4\sin^2 x + 1)dx$	1A	-1 if dx omitted
$= \int [2(1 - \cos 2x) + 1]dx$		
$= \int (3 - 2\cos 2x)dx$	1A	*No penalty for omitting brackets*
$= 3x - \sin 2x + c$	2A	- 1 if c omitted
sub $x = \frac{\pi}{2}$, $y = 0$	1M	
$c = \sin\pi - \frac{3}{2}\pi$		
$= -\frac{3\pi}{2}$	1A	
∴ the equation of the curve is		
$y = 3x - \sin 2x - \frac{3\pi}{2}$		
	6	
4. The two lines $\begin{cases} x + y = 4 \\ x - y = 2p \end{cases}$ intersect at		Alternatively: The 2 lines intersect at $x = p+2$
$(2+p,\ 2-p)$. [illegible]	1A	Area $= \int_0^{p+2} [(4-x)-(x-2p)]dx$ ..1+1A (limit, integrand)
They intersect the y-axis at (0, 4) and		$= [(4+2p)x - x^2]_0^{p+2}$
(0, −2p)	1A	$= (p+2)^2$ 1A
Area of △ $= \frac{\text{height} \times \text{base}}{2}$		etc.
$= \frac{1}{2}(2+p)(4+2p)$		
$= (p+2)^2$	1A	
$p^2 + 4p + 4 = 9$	1M	Accept ±9
$p^2 + 4p - 5 = 0$		
$(p+5)(p-1) = 0$		
$p = 1$ or -5	1+1A	
	6	

SOLUTION	MARKS	REMARKS
5. $\frac{d}{d\theta}\tan^3\theta = 3\tan^2\theta\sec^2\theta$	1A	
$\int \tan^2\theta\sec^2\theta\, d\theta = \frac{1}{3}\int d\tan^3\theta$		
$= \frac{1}{3}\tan^3\theta + c$	2A	-1 if c omitted still marked even if $\frac{1}{3}\tan^3\theta+C$ is not obtained by the specified method
$\int_0^{\frac{\pi}{3}} \tan^4\theta\, d\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta(\sec^2\theta - 1)d\theta$	2A	**Alternatively:** $\int_0^{\frac{\pi}{3}} \tan^2\theta\sec^2\theta\, d\theta$
$= \int_0^{\frac{\pi}{3}} \tan^2\theta\sec^2\theta\, d\theta - \int_0^{\frac{\pi}{3}} \tan^2\theta\, d\theta$		$= \int_0^{\frac{\pi}{3}} \tan^2\theta(1+\tan^2\theta)d\theta$ 2A
$= \int_0^{\frac{\pi}{3}} \tan^2\theta\sec^2\theta\, d\theta - \int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)d\theta$		
$= [\frac{1}{3}\tan^3\theta]_0^{\frac{\pi}{3}} - [\tan\theta - \theta]_0^{\frac{\pi}{3}}$ (for quo $\frac{1}{3}\tan^3\theta$)	1M+1A	$= \int_0^{\frac{\pi}{3}} (\sec^2\theta-1)d\theta + \int_0^{\frac{\pi}{3}} \tan^4\theta d\theta$
$= \sqrt{3} - \sqrt{3} + \frac{\pi}{3}$		
$= \frac{\pi}{3}$ (1.05)	1A	$\therefore \int_0^{\frac{\pi}{3}} \tan^4\theta d\theta$
	8	$= [\frac{\tan^3\theta}{3}]_0^{\frac{\pi}{3}} - [\tan\theta - \theta]_0^{\frac{\pi}{3}}$ ..1M+1A $= \frac{\pi}{3}$ 1A
6. $x^2 + y^2 - 2kx + 4ky + 6k^2 - 2 = 0$		$(x-k)^2 + (y+2k)^2 = 2 - k^2$
(a) radius $= \sqrt{k^2 + (2k)^2 - (6k^2-2)}$		
$= \sqrt{2 - k^2}$	1A	
$\sqrt{2 - k^2} > 1$	1M	
$\Rightarrow k^2 < 1$		
$\Rightarrow -1 < k < 1$	1A	
(b) Coordinates of the centre are $x = k$, $y = -2k$	1A	
$\therefore$ the locus of the centre lies on the line $2x + y = 0$	2A	
Since $-1 < k < 1$, we have $-1 < x < 1$ and $2 > y > -2$	1A for	either one of the inequalities or [with end-points $(-1, 2)$ and $(1, -2)$ excluded.]
$\therefore$ the locus is a line segment [with end-points $(-1, 2)$ and $(1, -2)$ excluded.]	1A	
	8	

SOLUTION	MARKS	REMARKS
7. (a) $\frac{1}{x^3}+\frac{3}{(2-3x)^2}=\frac{(2-3x)^2+3x^3}{x^3(2-3x)^2}$ $=\frac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}$	2A	
$\int_1^2 \frac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}\,dx$		
$=\int_1^2 [\frac{1}{x^3}+\frac{3}{(2-3x)^2}]\,dx$	1M	
$=\int_1^2 \frac{1}{x^3}\,dx+\int_1^2 \frac{3}{(2-3x)^2}\,dx$		
$=-\frac{1}{2}[\frac{1}{x^2}]_1^2+[\frac{1}{2-3x}]_1^2$	1+1A	
$=\frac{3}{8}+\frac{3}{4}$	1+1A	
$=\frac{9}{8}$		
	7	
(b) (i) Let $u=\sin\phi$, $du=\cos\phi\,d\phi$	1A	
$\int\frac{\cos\phi}{\sin^4\phi}\,d\phi=\int\frac{1}{u^4}\,du$	1A	
$=-\frac{1}{3u^3}+c$	2A	- 1 if omit c
$=-\frac{1}{3\sin^3\phi}+c$	1A	
	5	
(ii) Put $x=\tan\phi$, $dx=\sec^2\phi\,d\phi$	1A	
when $x=\frac{1}{\sqrt{3}}$, $\phi=\frac{\pi}{6}$; when $x=1$, $\phi=\frac{\pi}{4}$.	1A	
$\int_{\frac{1}{\sqrt{3}}}^{1}\frac{3\sqrt{1+x^2}}{x^4}\,dx$		
$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{3\sqrt{1+\tan^2\phi}}{\tan^4\phi}\sec^2\phi\,d\phi$	1M+1A	1M for limits, 1A for integrand
$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{3\sec^3\phi}{\tan^4\phi}\,d\phi=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{3\cos\phi}{\sin^4\phi}\,d\phi$	1A	no penality if ϕ is written as θ, etc
$=[-\frac{1}{\sin^3\phi}]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ by (i)	1M+1A	
$=\frac{1}{\sin^3\frac{\pi}{6}}-\frac{1}{\sin^3\frac{\pi}{4}}$		
$=\frac{1}{(\frac{1}{2})^3}-\frac{1}{(\frac{\sqrt{2}}{2})^3}$		
$=8-2\sqrt{2}$ (= 5.17)	1A	any figure roundable to 5.17
	8	

SOLUTION	MARKS	REMARKS
8. (a) $\sin 2\theta + \sin 8\theta = \sin 5\theta$		
$2\sin 5\theta \cos 3\theta = \sin 5\theta$	1A	
$\sin 5\theta\,(2\cos 3\theta - 1) = 0$		must be correct
$\sin 5\theta = 0$ or $\cos 3\theta = \frac{1}{2}$	1+1A	
$5\theta = n\pi$ or $3\theta = 2n\pi \pm \frac{\pi}{3}$		
$\therefore\ \theta = \frac{n\pi}{5}$ $(36n°)$	1A	
or $\frac{(6n\pm1)\pi}{9}$ $(120n°\pm20°)$, $n = 0, \pm1, \pm2, \ldots$	1A+1A	awarded if either one answer for θ is correct.
	6	
(b) $x = \frac{4\pi}{20}$, $y = 2.206$	1A	
$x = \frac{5\pi}{20}$, $y = 2.121$	1A	
Curve of $y = \sin x + 2\cos x$	3	Shape 2 curved line 1
(i) $5\sin x + 10\cos x = 11$		
$\Leftrightarrow \sin x + 2\cos x = 2.2$		
Consider the line $y = 2.2$	1A+1A	1A for equation 1A for line
The solutions are:		
$x = \frac{18\pi}{200}$ (or $\frac{19\pi}{200}$), $\frac{41\pi}{200}$ 0.267–0.299 0.628–0.660	1+1A	[illegible] decimal [illegible]
(ii) Consider the line $y = \frac{x}{4} + 2$	1A+1A	1A for equation 1A for line
$x = 0$, $y = 2.000$		
$x = \frac{5\pi}{20}$, $y = 2.196$		
The solutions are $x = 0$,	1A	
$\frac{44\pi}{200}$ $\left(\frac{45\pi}{200}\right)$ 0.675 – 0.707	2A	
	14	

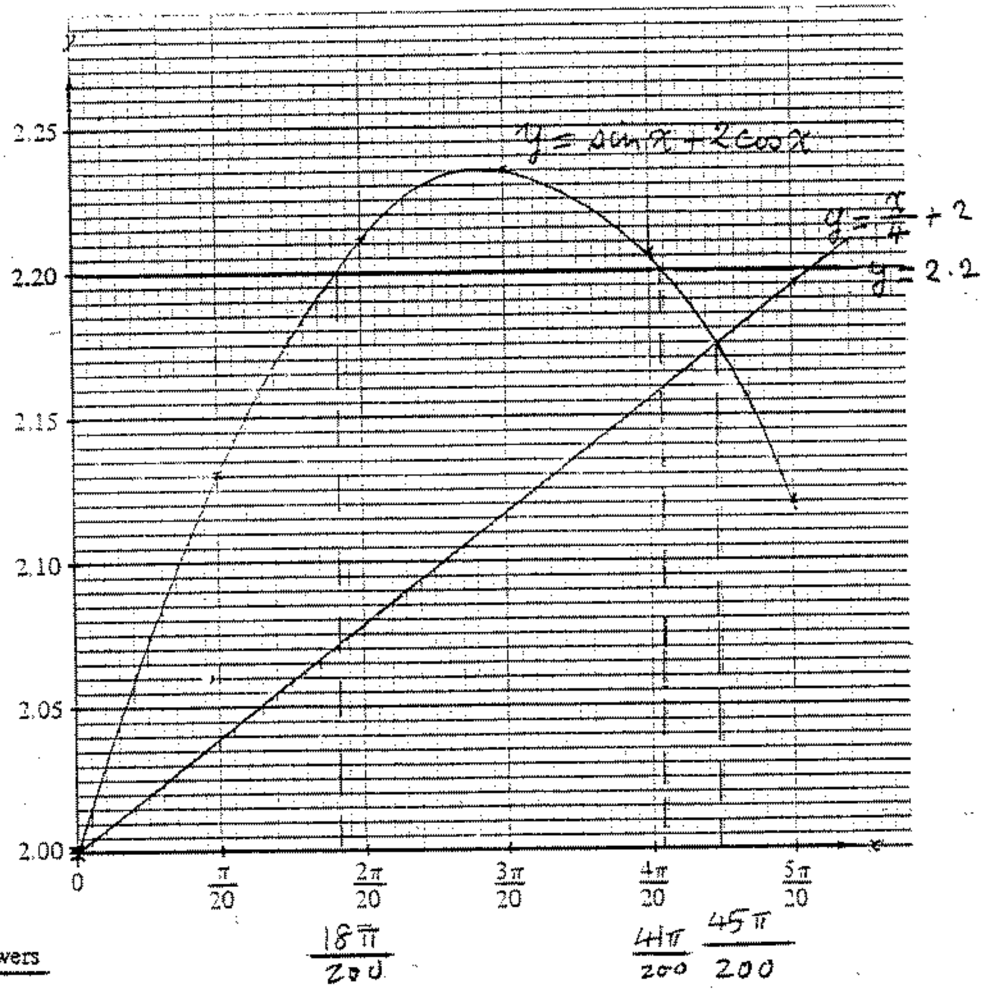
P5a.

Candidate Number	Centre Number	Seat Number	Total Marks on this page

8.(b) If you attempt this question, fill in the details in the first three boxes above and tie this sheet into your answer book.

Table 1

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$
$y = \sin x + 2\cos x$	2.000	2.132	2.211	2.236	2.206	2.121



Answers

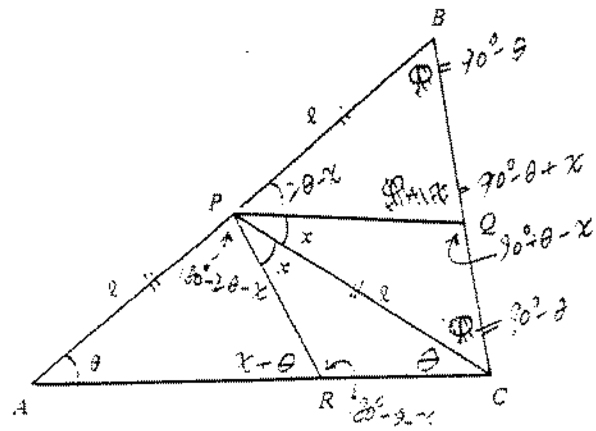
(i)

(ii)

SOLUTION	MARKS	REMARKS
9. (a) Equation of L is		
$y - 3 = m(x - 0)$ or $y = mx + 3$	1A	
Substituting in $x^2 + 4y^2 = 4$	1M	
$x^2 + 4(mx + 3)^2 = 4$		
$(4m^2+1)x^2 + 24mx + 32 = 0$	1A	
Discriminant $= (24m)^2 - 4(4m^2+1)32$	1M	
L cuts C at two real points iff		
$(24m)^2 - 4(4m^2+1)32 > 0$	1M	
$64m^2 - 128 > 0$		
$m^2 > 2$	1A	
$\therefore\ m > \sqrt{2}$ or $m < -\sqrt{2}$	~~1+1A~~ 2A	no mark for "and"; comma — C
If L touches C, $m = \pm\sqrt{2}$	1A	
Equations of tangents from P are $y = \sqrt{2}x + 3$ and $y = -\sqrt{2}x + 3$	1A	
	10	
(b) $2x + 8y\frac{dy}{dx} = 0$	1A	
$\frac{dy}{dx} = -\frac{x}{4y}$	1A	
At $(2\cos\theta, \sin\theta)$, gradient $= -\frac{\cos\theta}{2\sin\theta}$ $= -\frac{1}{2}\cot\theta$	1M+1A	
$\therefore$ the equation of the tangent T is $y - \sin\theta = -\frac{1}{2}\cot\theta(x - 2\cos\theta)$ or $x\cos\theta + 2y\sin\theta = 2$	1A	
Distance from P(0, 3) to the tangent is $d = \left\|\frac{6\sin\theta - 2}{\sqrt{\cos^2\theta + 4\sin^2\theta}}\right\| = \left\|\frac{6\sin\theta - 2}{\sqrt{3\sin^2\theta + 1}}\right\|$	2A	for any equivalent form; Abs. value optional
(i) when $\theta = \frac{3\pi}{2}$, $d = \left\|\frac{6(-1) - 2}{\sqrt{3(-1)^2 + 1}}\right\| = 4$	1A	Accept −4
(ii) when $\sin\theta = \frac{1}{3}$, $d = 0$	1A	
i.e. P lies on the tangent	1A	
	10	

SOLUTION	MARKS	REMARKS
10. (a) Put $x = a\sin\phi$, $dx = a\cos\phi\, d\phi$	1A	
when $x = a$, $\phi = \frac{\pi}{2}$; when $x = -a$, $\phi = -\frac{\pi}{2}$	1A	must be in radians
$\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx$ $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\phi}\,.\,a\cos\phi\, d\phi$		
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2\phi\, d\phi = \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1 + \cos2\phi)d\phi$	1A+1A	For integrands only
$= \frac{a^2}{2}\left[\phi + \frac{\sin2\phi}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ $= \frac{\pi a^2}{2}$	1A	
	5	
(b) (i) Equation of the circle is $x^2 + (y-b)^2 = a^2$ or $(y-b)^2 = a^2-x^2$	1A	
∴ equation of APB is $y - b = \sqrt{a^2 - x^2}$ or $y = b + \sqrt{a^2 - x^2}$	1A	
Equation of AQB is $y - b = -\sqrt{a^2 - x^2}$ or $y = b - \sqrt{a^2 - x^2}$	1A	
	3	
(ii) Volume = $\int_{-a}^{a}\pi(b + \sqrt{a^2-x^2})^2dx - \int_{-a}^{a}\pi(b - \sqrt{a^2-x^2})^2dx$	1+1M +1A	1M for $V = \int_{-a}^{a}\pi y^2dx$ 1M for " − " 1A for limits
$= \pi\int_{-a}^{a}[b^2 + 2b\sqrt{a^2-x^2} + (a^2-x^2)]dx - \pi\int_{-a}^{a}[b^2 - 2b\sqrt{a^2-x^2} + (a^2-x^2)]dx$		
$= \pi\int_{-a}^{a}4b\sqrt{a^2-x^2}\,dx$	1A	
$= 4\pi b \times \frac{\pi a^2}{2} = 2\pi^2a^2b$	1A	
	5	
(c) Volume $= 2\pi^2(2)^2(8) = 64\pi^2$ (mm³)	1A	Vol $= 64\pi^2$ (cm³) 1
$V = \int -32\pi^2(2-t)\,dt$	1M	$\int_{64\pi^2}^{V} dV = \int_0^t -32\pi^2(2-t)dt$ ......1M- ~~1A~~(limits 1M
$= 16\pi^2t^2 - 64\pi^2t + c$	1A	$V - 64\pi^2 = [-64\pi^2t+16\pi^2t^2]_0^t$ 1
When $t = 0$, $V = 64\pi^2$	1M	∴ $V = 16\pi^2t^2 - 64\pi^2t + 64\pi^2$ 1
∴ $c = 64\pi^2$	1A	
∴ $V = 16\pi^2t^2 - 64\pi^2t + 64\pi^2$		etc.
Putting $V = 0$, $16\pi^2(t^2 - 4t + 4) = 0$	1M	
$(t - 2)^2 = 0$, $t = 2$	1A	
∴ the piece of sweet dissolves completely in 2 hours		
	7	

SOLUTION	MARKS	REMARKS
11. (a) $PA = PC \Rightarrow \angle PCA = \theta$	1A	Alternatively:
$\therefore \angle PRA = x + \theta$	1A	$PA = PC \Rightarrow \angle PCA = \theta$ .... $\therefore \angle PRC = \pi - (x + \theta)$ ....
In $\triangle PRA$, $\dfrac{PR}{\sin\theta} = \dfrac{l}{\sin(x+\theta)}$	1M	In $\triangle PRC$, $\dfrac{PR}{\sin\theta} = \dfrac{l}{\sin(\pi-(x+\theta))}$ ....
$\therefore PR = \dfrac{l\sin\theta}{\sin(x+\theta)}$	1A	$\therefore PR = \dfrac{l\sin\theta}{\sin(x+\theta)}$ ....
	4	
(b) $PC = PB \Rightarrow \angle PCQ = \angle PBQ$ $(= \phi)$ (can be shown on diagram)	1A	Alternatively: $\angle PCQ = \angle PBQ$ ....
$\therefore \angle PQB = x + \phi$ $\quad \phi = \frac{\pi}{2} - \theta$	1A	$2(\theta + \phi) = \pi \Rightarrow \phi = \frac{\pi}{2} - \theta$ ......
In $\triangle PQB$, $\dfrac{PQ}{\sin\phi} = \dfrac{l}{\sin(x+\phi)}$	1M	(or $\triangle$ in semicircle) In $\triangle PCQ$,
$\therefore PQ = \dfrac{l\sin\phi}{\sin(x+\phi)} = \dfrac{l\cos\theta}{\cos(x-\theta)}$	1A	$\dfrac{PQ}{\sin(\frac{\pi}{2}-\theta)} = \dfrac{l}{\sin(\pi - x - (\frac{\pi}{2}-\theta))}$ .... $\therefore PQ = \dfrac{l\cos\theta}{\cos(x-\theta)}$ ....
	4	
(c) Area of $\triangle PQR = \frac{1}{2} PQ \cdot PR \sin 2x$	1M	
$= \dfrac{l^2 \sin\theta \cos\theta \sin 2x}{2\sin(x+\theta)\cos(x-\theta)}$	1A	
$= \dfrac{l^2}{2} \cdot \dfrac{\sin 2\theta \sin 2x}{\sin 2x + \sin 2\theta}$	1A	
$= \dfrac{l^2 \sin 2\theta}{2}\left(\dfrac{\sin 2x + \sin 2\theta - \sin 2\theta}{\sin 2x + \sin 2\theta}\right)$ $= \dfrac{l^2 \sin 2\theta}{2}\left(1 - \dfrac{\sin 2\theta}{\sin 2x + \sin 2\theta}\right)$ ..(*)	1A	working necessary
	4	



No penalty if 180° is written as π

SOLUTION	MARKS	REMARKS
11. (d) (i) Let $\theta = \frac{\pi}{8}$		
$\phi = \frac{\pi}{2} - \theta = \frac{3\pi}{8}$	1A	
$0 < x \leq \pi - 2\theta$ and $0 < x \leq \pi - 2\phi$	1A	
$0 < x \leq \frac{\pi}{4}$	1A	Accept $0 \leq x \leq \frac{\pi}{4}$, $x \leq \frac{\pi}{4}$
$0 < \sin 2x \leq 1$		
The maximum area of $\triangle PQR$ is		
$= \frac{l^2 \sin 2\theta}{2}\left(1 - \frac{\sin 2\theta}{1 + \sin 2\theta}\right)$	1M	Check candidate's range of x
$= \frac{l^2 \sin \frac{\pi}{4}}{2}\left(1 - \frac{\sin \frac{\pi}{4}}{1 + \sin \frac{\pi}{4}}\right)$		
$= \frac{l^2}{2(1+\sqrt{2})}$ $\left(\frac{l^2(\sqrt{2}-1)}{2}\right.$ or $\left.0.207 l^2\right)$	1A	Any figure roundable to $0.207 l^2$
(ii) If $\theta = \frac{\pi}{12}$, then		
$\phi = \frac{5\pi}{12}$ and $0 < x \leq \frac{\pi}{6}$	1A	
∴ the maximum area of $\triangle PQR$		
$= \frac{l^2 \sin \frac{\pi}{6}}{2}\left(1 - \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3} + \sin \frac{\pi}{6}}\right)$	1M	
$= \frac{l^2}{4}\left(1 - \frac{1}{\sqrt{3}+1}\right)$		5 for either (i) or (ii)
$= \frac{l^2 \sqrt{3}}{4(\sqrt{3}+1)}$ $\left(\frac{l^2 \sqrt{3}(\sqrt{3}-1)}{8}\right.$ or $\left.0.158 l^2\right)$	1A	3 for the other
	8	