10. A straight line through the point $R(-1, -1)$ has a variable slope m. It intersects the circle

$$x^2 + y^2 = 1$$

at A and B. Let P be the mid-point of AB.

(a) Find the coordinates of P in terms of m. (9 marks)

(b) The locus of P is a part of a curve C. Find the equation of C and name it. (6 marks)

(c) Sketch the locus of P. (5 marks)

11. (a) Show that $\dfrac{\sin 3\theta}{\sin\theta} = 2\cos 2\theta + 1$.

By putting $\theta = \frac{\pi}{4} + \phi$ in the above identity, show that

$$\frac{\cos 3\phi - \sin 3\phi}{\cos\phi + \sin\phi} = 1 - 2\sin 2\phi .$$

(7 marks)

(b) Using the substitution $\phi = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi = \int_{\frac{\pi}{2}}^{0} \frac{\sin 3u}{\cos u + \sin u}\, du .$$

Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi - \sin 3\phi}{\cos\phi + \sin\phi}\, d\phi .$$

(8 marks)

(c) Using the results in (a) and (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi .$$

(5 marks)

12. Let $f(x)$ be a function of x and let k and s be constants.

(a) By using the substitution $y = x + ks$, show that

$$\int_0^{s} f(x + ks)\, dx = \int_{ks}^{(k+1)s} f(x)\, dx .$$

Hence show that, for any positive integer n,

$$\int_0^{s} [f(x) + f(x+s) + \dots + f(x + (n-1)s)]\, dx = \int_0^{ns} f(x)\, dx .$$

(10 marks)

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}}$ by using the substitution $x = \sin\theta$.

Using this result together with (a), evaluate

$$\int_0^{\frac{1}{2n}} \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(x+\frac{1}{2n})^2}} + \frac{1}{\sqrt{1-(x+\frac{2}{2n})^2}} + \dots + \frac{1}{\sqrt{1-(x+\frac{n-1}{2n})^2}} \right) dx .$$

(10 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1984

附加數學 試卷一

ADDITIONAL MATHEMATICS PAPER I

8.30 am–10.30 am (2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

SECTION A (39 marks)
Answer ALL questions in this section.

1. Given $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j}$,

   $\overrightarrow{OB} = -\mathbf{i} + \mathbf{j}$.

   (a) Find the unit vector in the direction of $\overrightarrow{AB}$.

   (b) If P is a point such that $\overrightarrow{AP} = m\overrightarrow{AB}$, express $\overrightarrow{OP}$ in terms of m .

   (7 marks)

2. The surface area of a sphere is increasing at a rate of $8 \text{ cm}^2/\text{s}$. How fast is the volume of the sphere increasing when the surface area is $36\pi \text{ cm}^2$?

   (8 marks)

3. Let $z = (1 - 2i)^5$.

   (a) Using the binomial theorem, express z in the form $a + bi$, where a , b are real.

   (b) Find the real part of $\frac{1}{z}$.

   Hence write down the real part of $z + \frac{1}{z}$, correct to the nearest integer.

   (6 marks)

4. Solve for x :

   $$\left| 2|x| - 1 \right| \leqslant 2 .$$

   (5 marks)

5. Let α and β be the roots of the equation

   $$x^2 - 2x - (m^2 - m + 1) = 0 ,$$

   where m is a real number.

   (a) Show that $(\alpha - \beta)^2 > 0$ for any value of m .

   (b) Find the minimum value of $|\alpha - \beta|$.

   (7 marks)

6. ABC is a triangle in which $AB = AC$ and $\angle BAC = 2\theta$. The median $AD = h$. Find a point P on AD so that the product of the distances from P to the three sides of $\triangle ABC$ is a maximum.

   (6 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

7. In Figure 1, $ABCD$ is a square with $\overrightarrow{AB} = \mathbf{i}$ and $\overrightarrow{AD} = \mathbf{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.

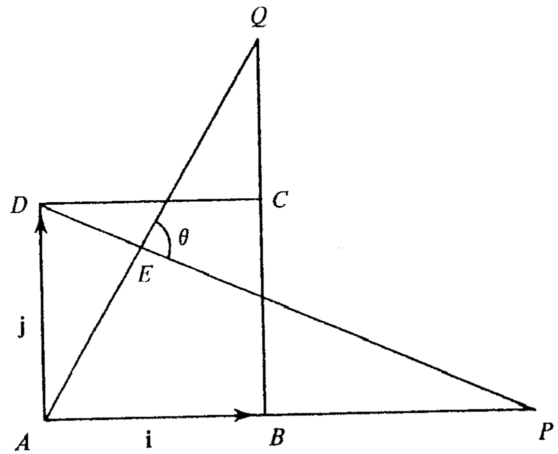


Figure 1

(a) By calculating $\overrightarrow{AQ} \cdot \overrightarrow{DP}$, find $\cos\theta$ in terms of m and k.

(8 marks)

(b) Given that $\dfrac{DE}{EP} = \dfrac{1}{4}$.

(i) Express $\overrightarrow{AE}$ in terms of k.

(ii) Let $\dfrac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m.

(iii) If $\theta = 90^\circ$, use the above results to find the values of k, m and r.

(12 marks)

8. Let $f(x) = 5x^2 + bx + c$, where b and c are real, $c > 0$ and $f(\frac{1}{2}) < 0$.

(a) Show that the equation

$$f(x) = 0$$

has two distinct real roots.

(6 marks)

(b) Let α and β $(\alpha < \beta)$ be the roots of $f(x) = 0$.

(i) By expressing $f(x)$ in factor form, show that

$$0 < \alpha < \frac{1}{2} < \beta .$$

(ii) If $\left|\alpha - \frac{1}{2}\right| = \left|\beta - \frac{1}{2}\right|$, find the value of b and hence the range of values of c.

(14 marks)

9. Let $\omega\ (\neq 1)$ be a cube root of 1.

(a) (i) Prove that $1 + \omega + \omega^2 = 0$.

(ii) Prove that for any integer k,

$$1 + \omega^{3k+1} + (\omega^2)^{3k+1} = 0 ,$$

$$1 + \omega^{3k+2} + (\omega^2)^{3k+2} = 0 .$$

(6 marks)

(b) Making use of the property of complex numbers: $|\alpha|^2 = \alpha\bar{\alpha}$, or otherwise, show that for any complex number z,

$$|1 - \omega\bar{z}| = |z - \omega| .$$

(5 marks)

(c) If z represents a variable point on the Argand diagram and c is a positive constant, what kind of curves does the equation

$$|1 - \omega\bar{z}| = c$$

represent ? Sketch the locus of z on the same diagram for each of the possible values of ω when $c = \frac{1}{2}$.

(9 marks)

10. In Figure 2, $ABCD$ is a square tin plate of side $2\sqrt{2}$ m. $PQRS$ is a square whose centre coincides with that of $ABCD$. The shaded parts are cut off and the remaining part is folded to form a right pyramid with base $PQRS$. Let $PQ = 2x$ metres and let the volume of the pyramid $= V$ cubic metres.

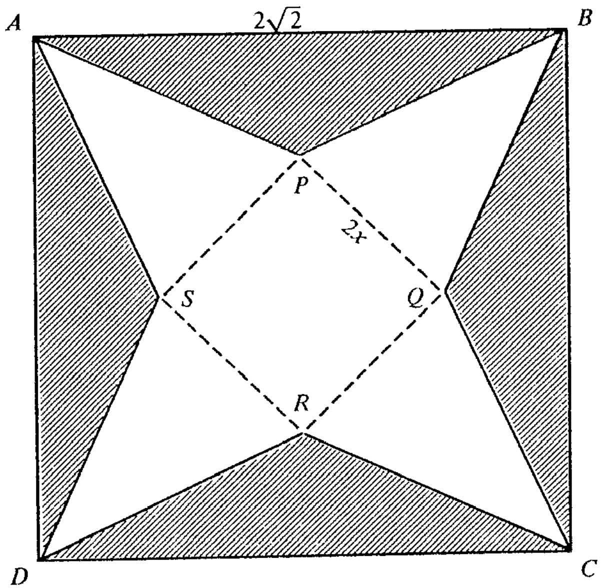


Figure 2

(a) Show that the height of the pyramid is given by $2\sqrt{1-x}$ metres.

Hence express V as a function of x.

(8 marks)

(b) Find the stationary points of the graph of V.

Find the equations of the tangents to the graph at the stationary points and at $x = 1$.

Hence sketch the graph for $0 \leqslant x \leqslant 1$.

(12 marks)

11. In Figure 3, AB is a railway 50 km long. C is a factory h kilometres from B such that $\angle ABC = 90°$. Goods are to be transported from C to A. The transportation cost per tonne of goods across the country by truck is \$2 per km, whereas by railway it is \$1 per km.

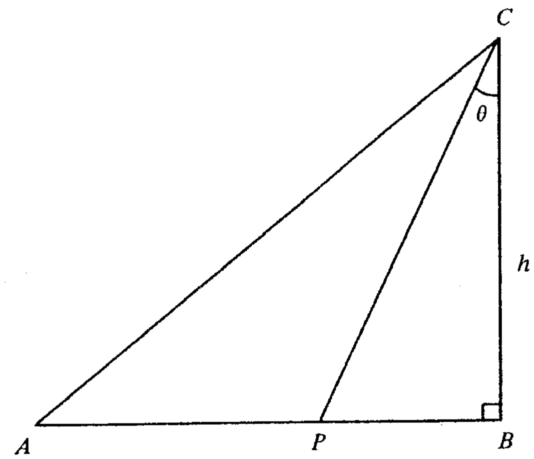


Figure 3

(a) Let P be a point on the railway, $\angle PCB = \theta$, and let \$$N$ be the total transportation cost for 1 tonne of goods from C to P and then to A. Find N in terms of θ and h.

(4 marks)

(b) If $h = 50$, show that the least transportation cost for 1 tonne of goods from C to A is \$$50(\sqrt{3}+1)$.

(7 marks)

(c) (i) Suppose $h > 50\sqrt{3}$. Show that $\tan\theta < \frac{1}{\sqrt{3}}$, and deduce that $\frac{dN}{d\theta} < 0$ for all possible values of θ.

(ii) If $h = 200$, what route should be taken so that the transportation cost is the least?

(9 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1984

附加數學 試卷二
ADDITIONAL MATHEMATICS PAPER II

11.15 am–1.15 pm (2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

SECTION A (39 marks)
Answer ALL questions in this section.

1. In the expansion of $\left(x^2 + \frac{a}{x}\right)^8$, where $a \neq 0$, the coefficient of x^r is denoted by B_r. Find the value of a if $B_7 = 4B_{10}$.

(5 marks)

2. Prove by mathematical induction that, for all positive integers n, $4n^3 - n$ is divisible by 3.

(6 marks)

3. The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = 4\sin^2 x + 1 .$$

If the curve cuts the x-axis at $x = \frac{\pi}{2}$, find the equation of the curve.

(6 marks)

4. The area of the triangle bounded by the two lines $x + y = 4$ and $x - y = 2p$ and the y-axis is 9. Find the two values of p.

(6 marks)

5. Making use of the derivative of $\tan^3\theta$, find

$$\int \tan^2\theta \sec^2\theta \, d\theta .$$

Hence evaluate $\int_0^{\frac{\pi}{3}} \tan^4\theta \, d\theta$

(8 marks)

6. Given the equation $x^2 + y^2 - 2kx + 4ky + 6k^2 - 2 = 0$.

(a) Find the range of values of k so that the equation represents a circle with radius greater than 1.

(b) Find the locus of the centre of the circle as k varies within the range in (a).

(8 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

7. (a) Prove that $\frac{1}{x^3} + \frac{3}{(2-3x)^2} = \frac{3x^3 + 9x^2 - 12x + 4}{9x^5 - 12x^4 + 4x^3}$.

Hence find the value of $\int_1^2 \frac{3x^3 + 9x^2 - 12x + 4}{9x^5 - 12x^4 + 4x^3} \, dx$.

(7 marks)

(b) (i) Using the substitution $u = \sin\phi$, find $\int \frac{\cos\phi}{\sin^4\phi} \, d\phi$.

(ii) Using the substitution $x = \tan\phi$ and the result of (i), evaluate

$$\int_{\frac{1}{\sqrt{3}}}^{1} \frac{3\sqrt{1+x^2}}{x^4} \, dx .$$

(13 marks)

8. **If you attempt this question, you should refer to the separate supplementary leaflet provided.**

(a) Find the general solution of the equation

$$\sin 2\theta + \sin 8\theta = \sin 5\theta .$$

(6 marks)

(b) Let $y = \sin x + 2\cos x$. Complete Table 1 on the separate answer sheet provided and use the data to plot the graph of

$$y = \sin x + 2\cos x .$$

By adding two suitable straight lines to the graph, find the solutions of the equations

(i) $5\sin x + 10\cos x = 11$,

(ii) $\sin x + 2\cos x = \frac{x}{4} + 2$.

Give your answers correct to the nearest $\frac{\pi}{200}$.

(14 marks)

9. Given the curve $C : x^2 + 4y^2 = 4$ and the point $P(0, 3)$.

(a) L is a line of variable slope m through P. If L cuts C at two distinct real points, find the possible range of values of m.

If L touches C, what are the possible values of m ?

Hence write down the equations of the two tangents from P to C.

(10 marks)

(b) $Q(2\cos\theta, \sin\theta)$ is a point on C. Find by differentiation the gradient of C at Q and hence show that the equation of the tangent T at Q is

$$x\cos\theta + 2y\sin\theta = 2 .$$

Express the distance from P to the tangent T in terms of θ.

Find the distance when

(i) $\theta = \frac{3\pi}{2}$,

(ii) $\sin\theta = \frac{1}{3}$.

Interpret case (ii) geometrically.

(10 marks)

10. (a) Use the substitution $x = a\sin\phi$ to show that

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{2} .$$

(5 marks)

(b) Figure 1 shows two semicircles APB and AQB with a common centre $C(0, b)$ and equal radii a. AB is parallel to the x-axis.

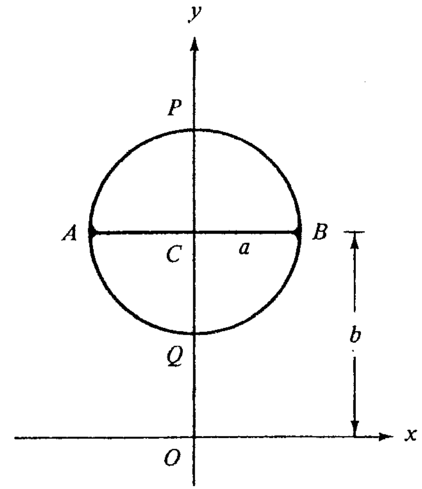


Figure 1

(i) Show that the equation of APB is

$$y = b + \sqrt{a^2 - x^2}$$

and that of AQB is

$$y = b - \sqrt{a^2 - x^2} .$$

(ii) The region bounded by the two semicircles is revolved about the x-axis to generate a solid (called an anchor-ring). Use the result in (a) to prove that the volume of the anchor-ring is $2\pi^2 a^2 b$.

(8 marks)

(c) A sweet has the form of an anchor-ring with $a = 2$ mm and $b = 8$ mm. Write down its volume in terms of π.

The sweet is now dropped into water and it dissolves with a rate of change of volume given by

$$\frac{dV}{dt} = -32\pi^2(2-t) \text{ mm}^3/\text{h},$$

where V is the volume in mm^3, t is the time in hours.

Find V in terms of t and hence find the time required to dissolve the whole sweet completely.

(7 marks)

11. In Figure 2, ABC is a triangle with $\angle A = \theta$. P is a point on AB such that $PA = PB = PC = \ell$. R and Q are points on AC and BC, respectively, such that $\angle QPC = \angle RPC = x$.

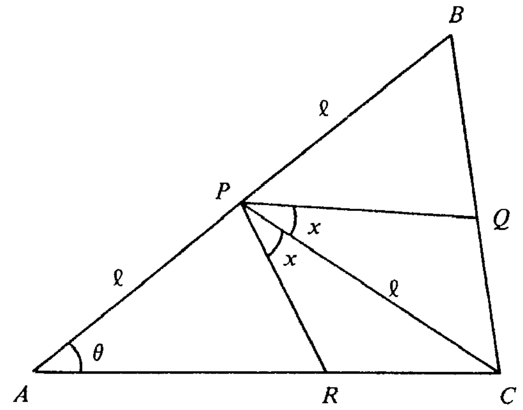


Figure 2

(a) Show that $PR = \dfrac{\ell \sin\theta}{\sin(x+\theta)}$.

(4 marks)

(b) Find $\angle PCQ$ in terms of θ and hence find PQ in terms of ℓ, x and θ.

(4 marks)

(c) Show that the area of $\Delta PQR = \dfrac{\ell^2 \sin\theta\cos\theta\sin 2x}{2\sin(x+\theta)\cos(x-\theta)}$,

and show that it can be expressed as

$$\frac{\ell^2\sin 2\theta}{2}\left(1 - \frac{\sin 2\theta}{\sin 2x + \sin 2\theta}\right) \quad \text{..........................}(*)$$

(4 marks)

(d) (i) If $\theta = \frac{\pi}{8}$, find the possible range of values of x.

Hence use $(*)$ to deduce the maximum area of ΔPQR and express it in terms of ℓ.

(ii) If $\theta = \frac{\pi}{12}$, what is the possible range of values of x?

Express the maximum area of ΔPQR in terms of ℓ.

(8 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1984

附加數學 試卷二(附頁)

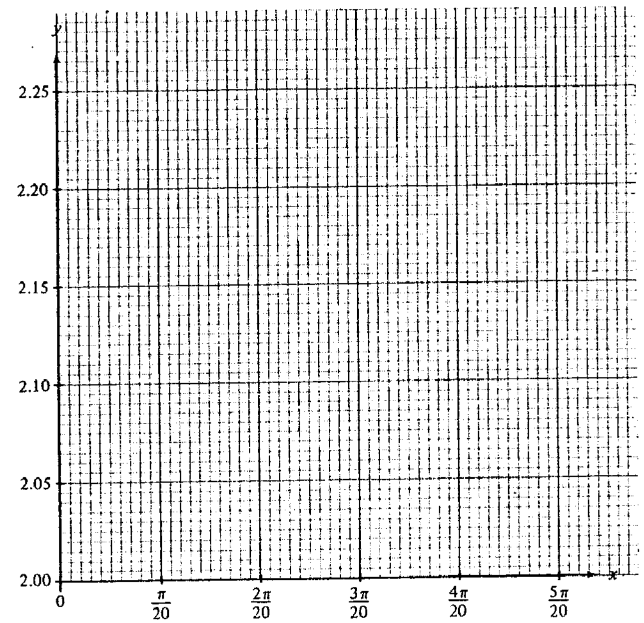
ADDITIONAL MATHEMATICS PAPER II
(SUPPLEMENTARY LEAFLET)

Candidate Number	Centre Number	Seat Number	Total Marks on this page

8.(b) If you attempt this question, fill in the details in the first three boxes above and tie this sheet into your answer book.

Table 1

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$
$y = \sin x + 2\cos x$	2.000	2.132	2.211	2.236		



Answers

(i)

(ii)

1984

Additional Mathematics I

1. (a) $-\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

   (b) $(3 - 4m)\mathbf{i} + (3m - 2)\mathbf{j}$

2. 12 cm^3/s

3. (a) $41 + 38i$

   (b) $\text{Re}(\frac{1}{z}) = \frac{41}{3125}$

   $\text{Re}(z + \frac{1}{z}) = 41$

   (correct to the nearest integer)

4. $-\frac{3}{2} \leqslant x \leqslant \frac{3}{2}$

5. (b) $\sqrt{7}$

6. $AP = \frac{2}{3}h$

7. (a) $\dfrac{k - m}{\sqrt{(m^2 + 2m + 2)(k^2 + 2k + 2)}}$

   (b) (i) $\frac{1+k}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

   (ii) $r\mathbf{i} + r(1 + m)\mathbf{j}$

   (iii) $r = \frac{2}{5}$

   $m = k = 1$

8. (b) (ii) $b = -5$

   $0 < c < \frac{5}{4}$

10. (a) $V = \frac{8}{3}x^2\sqrt{1 - x}$

    (b) $(0, 0)$, $(\frac{4}{5}, \frac{128}{75\sqrt{5}})$

    $V = 0$, $V = \frac{128}{75\sqrt{5}}$, $x = 1$

11. (a) $N = 2h\sec\theta + (50 - h\tan\theta)$

    (c) (ii) Goods should be transported directly from C to A by truck.

1984

Additional Mathematics II

1. 2

3. $y = 3x - \sin 2x - \frac{3\pi}{2}$

4. $p = 1$ or -5

5. $\frac{1}{3}\tan^3\theta + c$

   $\frac{\pi}{3}$

6. (a) $-1 < k < 1$

   (b) The locus is a line segment with end-points $(-1, 2)$ and $(1, -2)$ excluded.

7. (a) $\frac{9}{8}$

   (b) (i) $-\frac{1}{3\sin^3\phi} + c$

   (ii) $8 - 2\sqrt{2}$

8. (a) $\theta = \frac{n\pi}{5}$ or $\frac{(6n \pm 1)\pi}{9}$,

   $n = 0, \pm 1, \pm 2, \ldots$

   (b) $x = \frac{4\pi}{20}$, $y = 2.206$

   $x = \frac{5\pi}{20}$, $y = 2.121$

   (i) $\frac{18\pi}{200}$, $\frac{41\pi}{200}$

   (ii) 0, $\frac{44\pi}{200}$

9. (a) $m > \sqrt{2}$ or $m < -\sqrt{2}$

   $m = \pm\sqrt{2}$

   $y = \sqrt{2}x + 3$, $y = -\sqrt{2}x + 3$

   (b) $d = \left|\dfrac{6\sin\theta - 2}{\sqrt{3\sin^2\theta + 1}}\right|$

   (i) 4

   (ii) 0, P lies on the tangent

10. (c) $64\pi^2$ mm^3

    $V = 16\pi^2 t^2 - 64\pi^2 t + 64\pi^2$

    2 hours

11. (b) $\angle PCQ = \frac{\pi}{2} - \theta$

    $PQ = \dfrac{\ell\cos\theta}{\cos(x - \theta)}$

    (d) (i) $0 < x \leqslant \pi - 2\theta$ and

    $0 < x \leqslant \pi - 2\phi$

    Maximum area $= \dfrac{\ell^2}{2(1 + \sqrt{2})}$

    (ii) $0 < x \leqslant \frac{\pi}{6}$

    Maximum area $= \dfrac{\ell^2\sqrt{3}}{4(\sqrt{3} + 1)}$