進佳數學團隊 **– Dr. Herbert Lam** 林康榮博士
HKDSE Mathematics M2

A. Measures of Angles

It is known that in daily life, angles are measured in degrees. In mathematics, it is customary to measure angles in radians. The radian measure of an angle θ whose vertex is at the center of a circle is defined as the following ratio

$$\theta = \frac{s}{r},$$

where s is the length of the intercepted arc, and r is the radius, as shown in Figure 1.
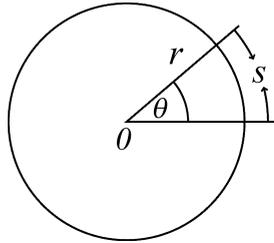
Figure 1

Hence, we have the following formulas concerning a sector with radius r and angle θ (measured in radian).

   (i)   Arc length $s = r\theta$,

   (ii) Area $= \frac{1}{2} r^2 \theta$.

It follows that

$$360° = 2\pi \text{ radians},$$
$$180° = \pi \text{ radians},$$
$$1° = \frac{2\pi}{360} \simeq 0.01745329 \text{ radians},$$
$$1 \text{ radian} = \frac{360°}{2\pi} \simeq 57°17'44.8''.$$

Moreover, $30° = \frac{\pi}{6}$ radians, $45° = \frac{\pi}{4}$ radians, $60° = \frac{\pi}{3}$ radians,

$$90° = \frac{\pi}{2} \text{ radians, etc.,}$$

are frequently used in mathematics.

Example 1

A piece of paper in the form of a sector is folded to make a cup,
which takes the form of a circular cone.
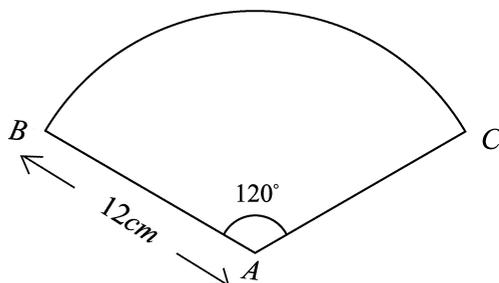
Figure 2
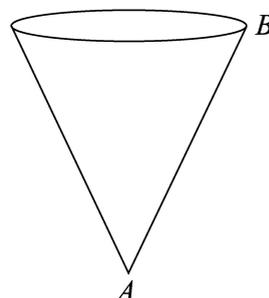
Figure 3

Find the capacity of the cup.

[Solution]

The arc length of $\overset{\frown}{BC} = AB \cdot \angle BAC$

$$= 12 \cdot \frac{2\pi}{3} \quad (120° = \frac{2\pi}{3} \text{ rad})$$

$$= 8\pi \ cm.$$

Let r be the base radius of the cone.

Then $\qquad 2\pi r = 8\pi$,

$\qquad\qquad r = 4 \ cm$.

Volume of the cone, $V = \frac{1}{3}\pi r^2 h$

Figure 4

With $r = 4$, $h = \sqrt{12^2 - 4^2}$ (see vertical section of the cone in

$\qquad\qquad\qquad\qquad\qquad$ Figure 4)

$$= 8\sqrt{2}.$$

$\therefore \ V = \frac{1}{3} \cdot 4^2 \cdot 8\sqrt{2}\pi$

$$= \frac{128\sqrt{2}}{3}\pi,$$

i.e. capacity of the cone is $\dfrac{128\sqrt{2}}{3}\pi \ \ c.c.$
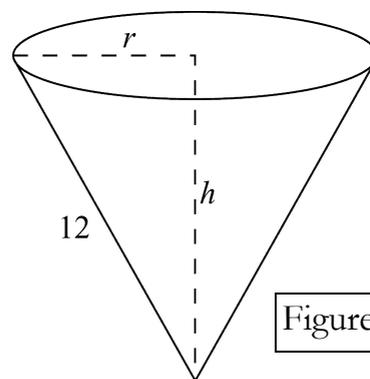
B. Fundamentals of
Trigonometric Functions

Let $P(x, y)$ be a point on the circle with centre at the origin and radius r, and let θ be the angle measured from the x-axis to OP, in the counter-clockwise direction.
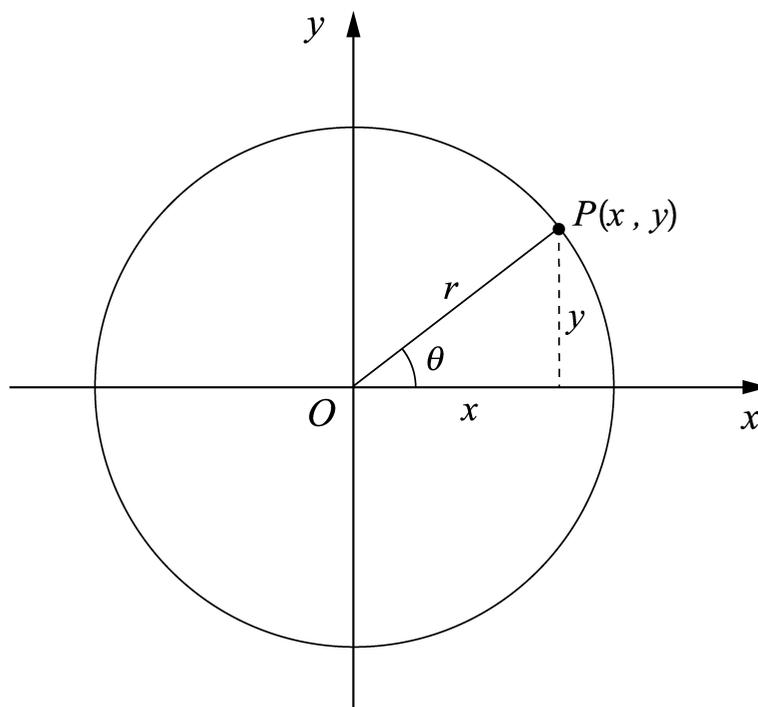
Figure 1

With reference to Figure 1, the six **trigonometric ratios** (also known as **trigonometric functions**) of θ, are defined as follows.

Sine of θ	$\sin \theta = \dfrac{y}{r}$
Cosine of θ	$\cos \theta = \dfrac{x}{r}$
Tangent of θ	$\tan \theta = \dfrac{y}{x}$
Cotangent of θ	$\cot \theta = \dfrac{x}{y}$
Secant of θ	$\sec \theta = \dfrac{r}{x}$
Cosecant of θ	$\csc \theta = \dfrac{r}{y}\ *$

Table 1

* also written as $\operatorname{cosec}\theta$, but $\csc\theta$ is used throughout this book.

Note:

In some elementary texts, trigonometric ratios of an angle are defined as the ratios between the sides of a right triangle, *i.e.*

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}.$$
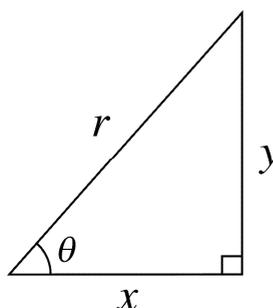
Figure 2

The above definitions are valid for the acute angle θ. If θ is greater than or equal to $90°$, the trigonometric ratios of θ are "redefined", as shown in Figure 1 and Table 1.

Obviously, we have the following relationship:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \ , \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

and we can easily verify the following identities:

$$\sin^2\theta + \cos^2\theta = 1,$$

$$1 + \tan^2\theta = \sec^2\theta,$$

$$1 + \cot^2\theta = \csc^2\theta.$$

Example 1

Show that if θ is sufficiently small, then $\sin\theta \approx \theta$.

[Solution]

This approximation comes from the well known limit,

$\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, which implies $\dfrac{\sin\theta}{\theta} \approx 1$.

This can be established informally, from a geometric viewpoint. Suppose a regular polygon of n sides is inscribed in a circle of radius r.
We can visualize that the area of the polygon will be close to the area of the circle as the number of sides increases. Consider a portion of the polygon, a triangle inscribed in a sector as shown in Figure 3.

Figure 3

Area of $\triangle OAB = \dfrac{1}{2}OB \cdot AC$

$$= \frac{1}{2}OB \cdot OA \, \sin\theta$$

$$= \frac{1}{2}r^2\sin\theta.$$
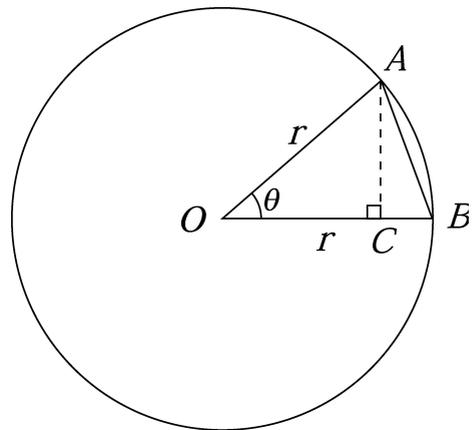
Since the polygon consists of n such triangles, its area is

$$n \cdot \frac{1}{2}r^2\sin\theta = \frac{n}{2}r^2\sin\frac{2\pi}{n}.$$

The area of the circle is known to be πr^2.

Thus $\qquad \dfrac{n}{2} r^2 \sin \dfrac{2\pi}{n} \approx \pi r^2 \qquad$ as n gets large, or equivalently, θ becomes very small.

This implies $\qquad \dfrac{\sin \dfrac{2\pi}{n}}{\dfrac{2\pi}{n}} \approx 1$,

i.e. $\sin\theta \approx \theta$ if $\theta = 2\pi/n$ is close to zero.

From another viewpoint, the area of ΔOAB and the area of sector OAB are approximately equal if θ is very small.

If θ cannot be expressed as $2\pi/n$, we can find $\theta \approx \theta'$ so that $\theta' = 2\pi/n$ for some n.

1. Special angles

The following trigonometric ratios are frequently used in regard to the two special types of right triangles.

(a) $\quad \sin 30° = \cos 60° = \dfrac{1}{2}$,

$\quad \sin 60° = \cos 30° = \dfrac{\sqrt{3}}{2}$,

$\quad \tan 30° = \dfrac{1}{\sqrt{3}}$,

$\quad \tan 60° = \sqrt{3}$.
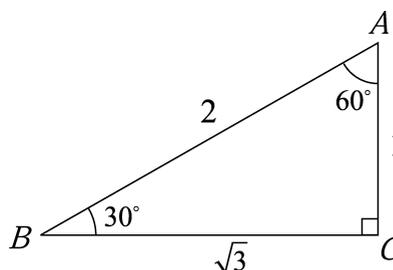
Figure 4

(b) $\quad \sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$ or $\dfrac{\sqrt{2}}{2}$,

$\quad \tan 45° = 1$.
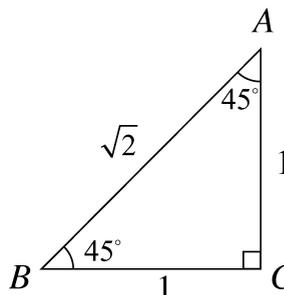
Figure 5

Furthermore,

$\sin 0° = \tan 0° = \cos 90° = 0$,

$\sin 90° = \cos 0° = 1$,

$\tan 90°$ is undefined. (also denoted by ∞, but formally we should write

$$\lim_{\theta \to \frac{\pi}{2}} \tan\theta = \infty \quad \text{for} \quad 0 \le \theta < \frac{\pi}{2})$$

These are summarized in Table 2

| θ	radian	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
	degree	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos\theta$		1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0
$\sin\theta$		0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
$\tan\theta$		0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Table 2

2. Signs of trigonometric functions in the four quadrants

The signs of these trigonometric functions are determined by the signs of x and y in each respective quadrant, and are summarized in Table 3.

| Quadrant	x	y	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+	+	+
II	−	+	+	−	−	−	−	+
III	−	−	−	−	+	+	−	−
IV	+	−	−	+	−	−	+	−

Table 3

It is easier to memorize above using the **CAST** rule, as shown in Figure 6.
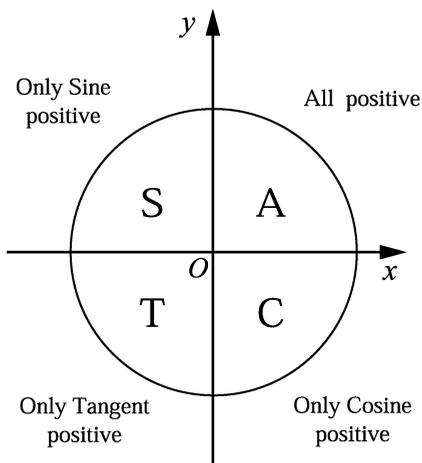
Special case:

If $r = 1$, the six trigonometric functions can be shown in Figure 7.
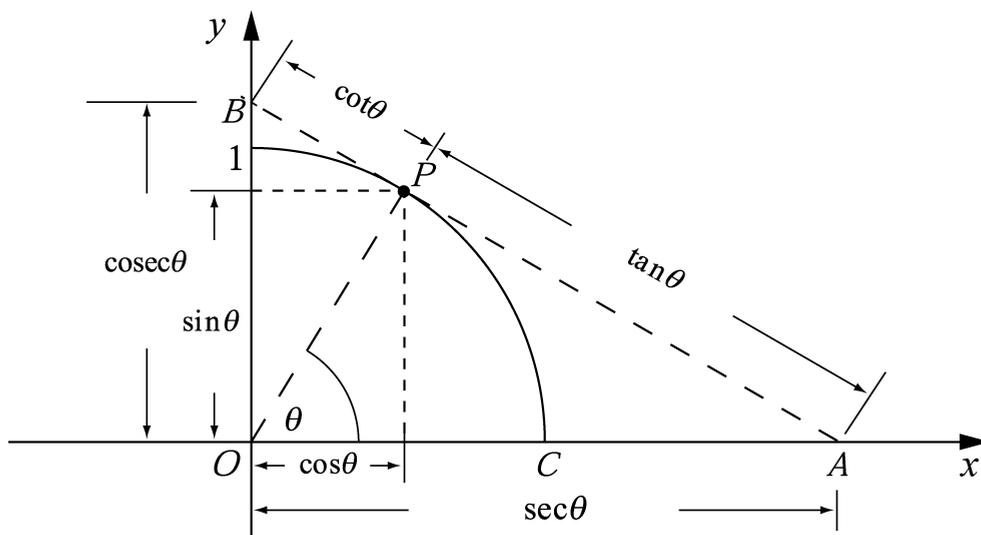
Figure 7

$\sin\theta$ and $\cos\theta$ are obvious since $OP = r = 1$. The others can be easily verified, *e.g.*

in ΔOPA, $\tan\theta = \dfrac{AP}{OP}$, $OP = 1 \Rightarrow AP = \tan\theta$,

$$\sec\theta = \frac{OA}{OP} \Rightarrow OA = \sec\theta.$$

Similarly, take ΔOPB to obtain $\cot\theta$ & $\csc\theta$.

Example 2

Given that $\dfrac{\cos^2 \theta}{1 + 3\sin^2 \theta} = \dfrac{4}{13}$, where $0 \le \theta \le 180°$,

Find the values of $\dfrac{\cos \theta}{1 + 3\sin \theta}$.

[Solution]

Cross multiplying $\dfrac{\cos^2 \theta}{1 + 3\sin^2 \theta} = \dfrac{4}{13}$,

we obtain

$$13\cos^2 \theta = 4 + 12\sin^2 \theta.$$

Reducing and solving,

$$13\cos^2 \theta = 4 + 12\left(1 - \cos^2 \theta\right)$$

$$25\cos^2 \theta = 16$$

$$\cos^2 \theta = \frac{16}{25} \Rightarrow \cos \theta = \pm\frac{4}{5},$$

$$\sin^2 \theta = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow \sin \theta = \frac{3}{5}. \quad (\because \ 0 \le \theta \le 180°)$$

Hence, $\dfrac{\cos \theta}{1 + 3\sin \theta} = \pm\dfrac{\dfrac{4}{5}}{1 + 3 \cdot \dfrac{3}{5}}$

$$= \pm\frac{4}{14} = \pm\frac{2}{7}.$$

3. Trigonometric functions in special situations

(a) $180° \pm \theta$, $360° - \theta$ (the latter is also known as $-\theta$)

(b) $90° \pm \theta$, $270° \pm \theta$

These are summarized in Table 4 & 5.

angle trig. Function	$-\theta$	$\pi - \theta$ $180° - \theta$	$\pi + \theta$ $180° + \theta$	$2\pi - \theta$ $360° - \theta$
sin	$-\sin\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$
cos	$\cos\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$

Table 4

| angle trig. function	$\pi/2 - \theta$ $90° - \theta$	$\pi/2 + \theta$ $90° + \theta$	$3\pi/2 - \theta$ $270° - \theta$	$3\pi/2 + \theta$ $270° + \theta$
sin	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
tan	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

Table 5

C. Graphs of
Trigonometric Functions

The angle θ is measured from the x-axis, and is taken positive in the counterclockwise direction. The graphs of the functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ are given in Figures 1 to 6.
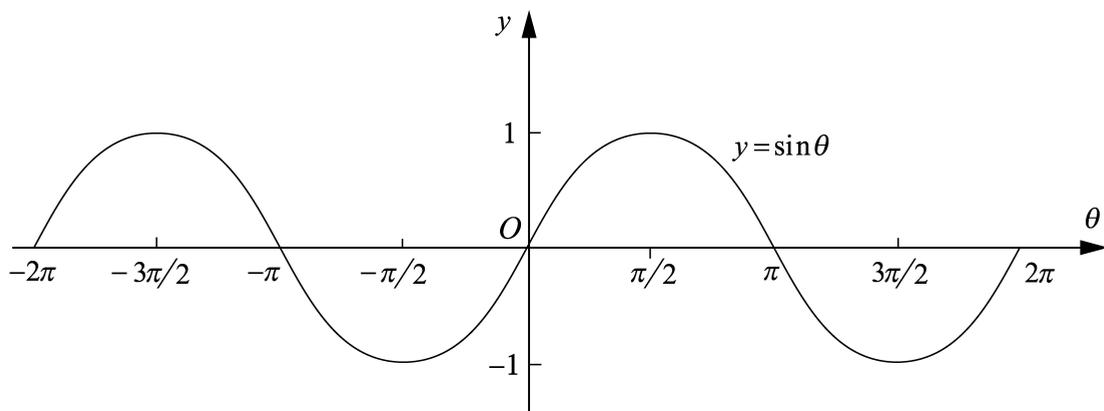
Figure 1
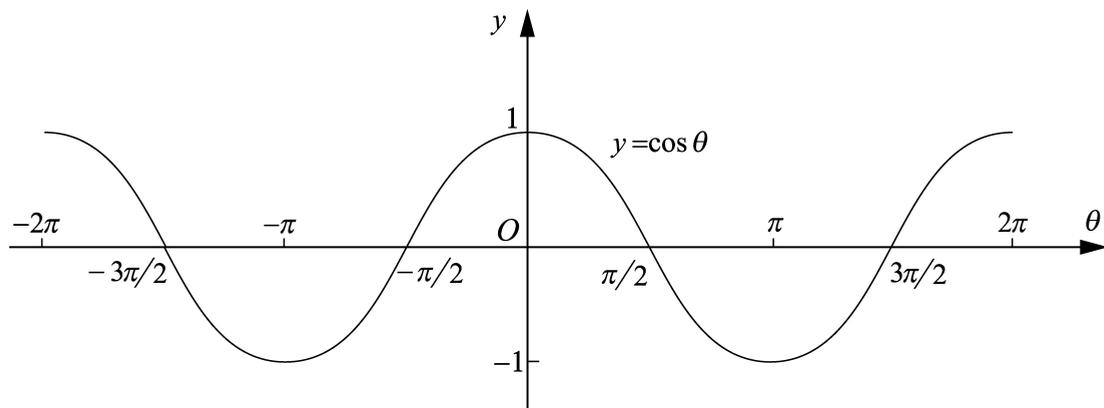
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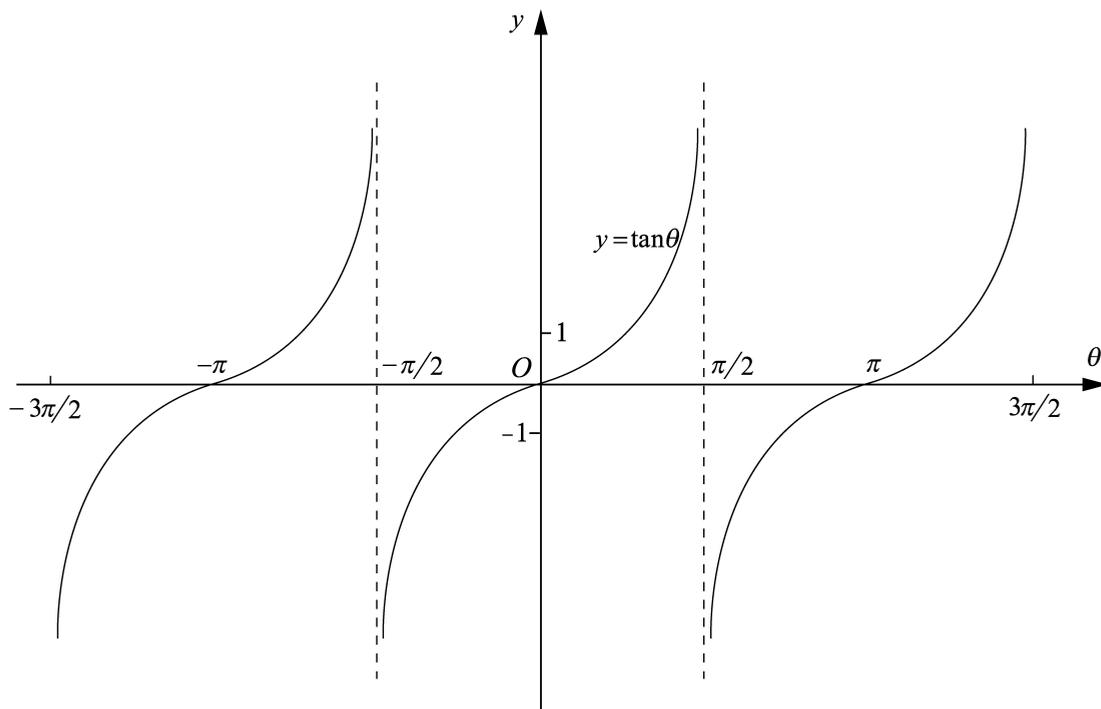
Figure 3
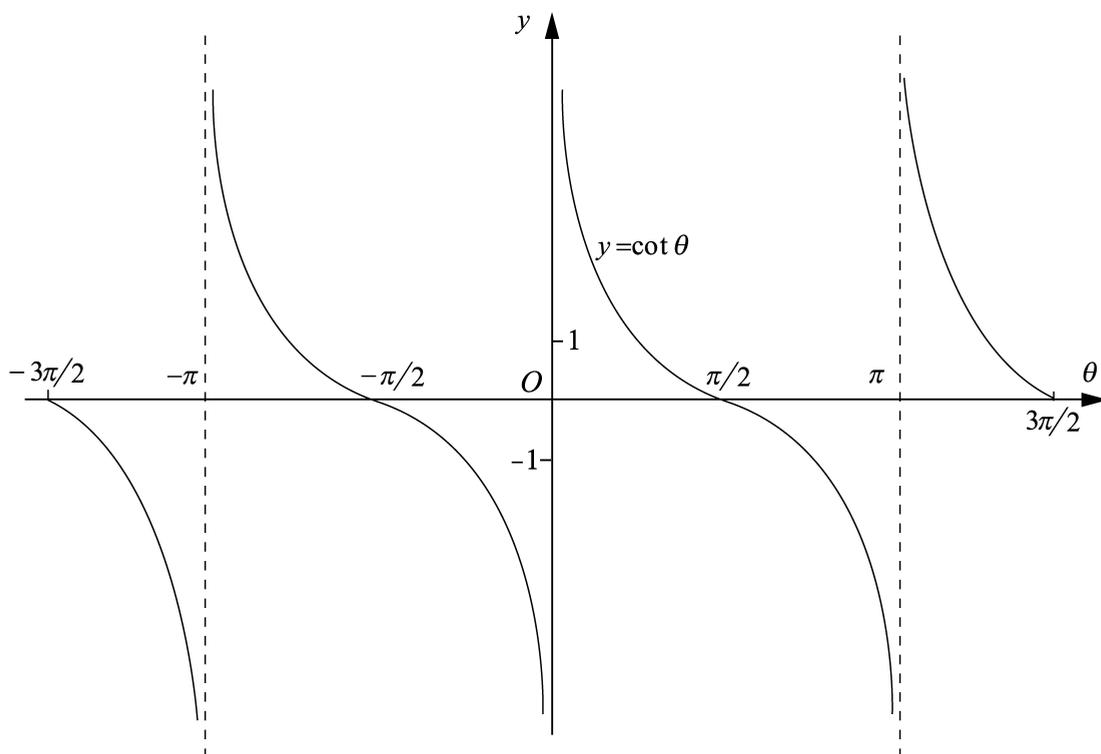
Figure 4
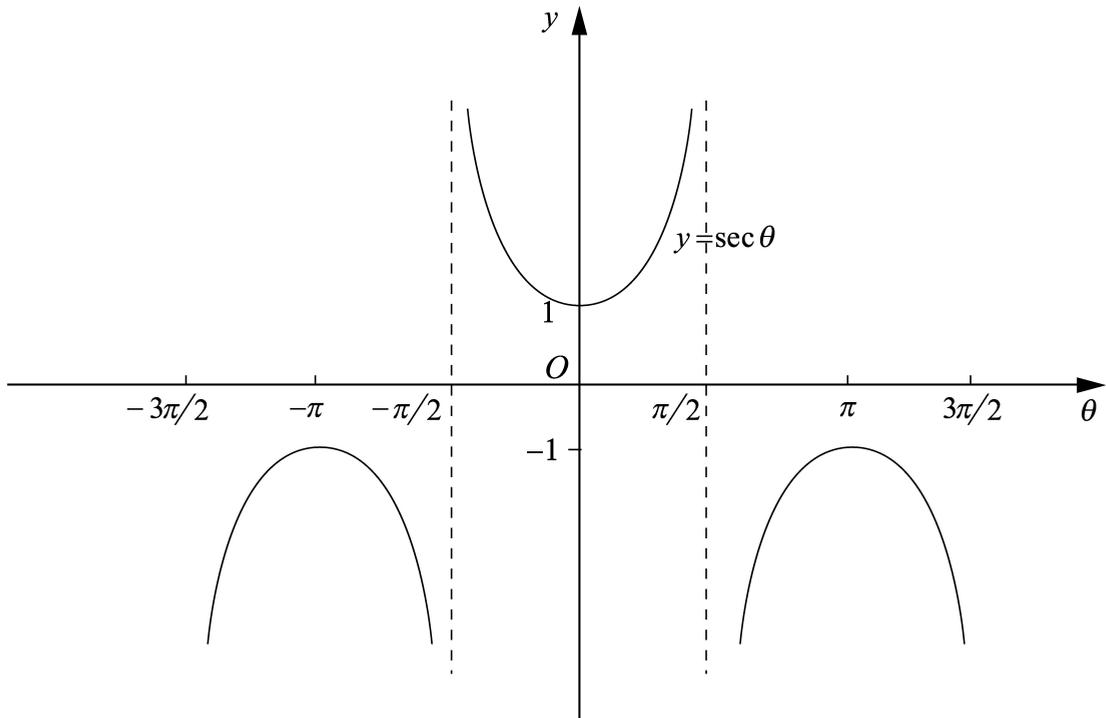
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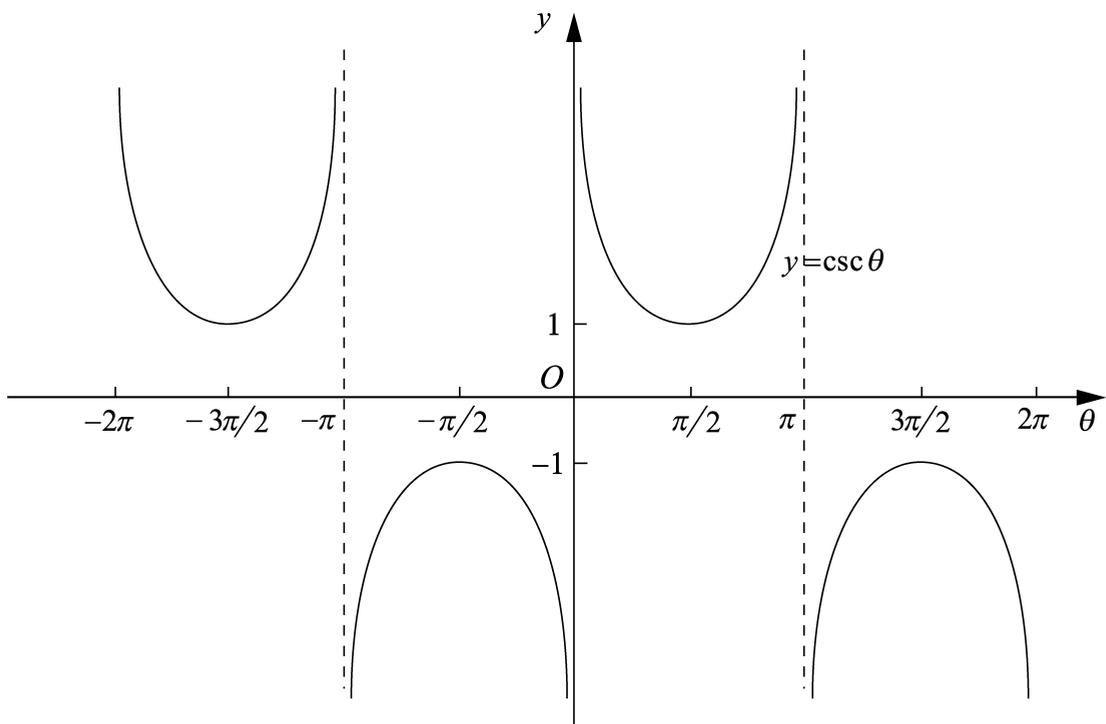
Figure 6

Note:

The sine and cosine functions are frequently used in physics. They are of the form:

$$y = a \, \sin(\omega t + \alpha),$$

where a is the amplitude,

$\dfrac{2\pi}{\omega}$ is the period,

α is the phase angle.

The graph is depicted below.

Figure 7

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