A. Functions

In mathematics, any rule that assigns to each element in one set some elements from another set is called a **function**. The sets may be sets of numbers, sets of points or sets of objects of any kind. The sets do not have to be the same.

1. Domain and Range

A function(mapping) f from a set A to a set B is a rule that assigns a single element of B to each element in A. Usually, such a function is denoted by $f: A \to B$.

Euler invented a symbolic way to express "y is a function of x" by writing $y = f(x)$, which we read "y equals f of x". The set A is called the **domain** of f while the set $\{ y : y \in B \text{ and } y = f(x) \text{ for some } x \in A \}$ is called the **range** of f. The element y is said to be the image of x and x is the pre-image of y. Conventionally, x and y are called the **independent variable** and **dependent variable** respectively.

(a) Examples of functions (each domain is taken to be the largest set of real x values for which the formula gives real y values)

Function	Domain	Range	Remark
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$	The formula gives a positive real y value for any real number x.
$y = \sqrt{1-x^2}$	$[-1,1]$	$[0,1]$	Beyond this domain, the quantity $1-x^2$ is negative and its square roots is not a real number.
$y = \frac{1}{x}$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	The formula gives a real y value for every x except $x=0$.
$y = \sqrt{x}$	$[0,\infty)$	$[0,\infty)$	The formula gives a real y value only when x is positive or zero.
$y = \sqrt{4-x}$	$(-\infty,4]$	$[0,\infty)$	The formula gives a real y value only when the quantity $4-x$ cannot be negative, i.e. $x \le 4$.

Example 1

Consider the graphs depeited in figures below. Show that f and g are functions but h and k are not.

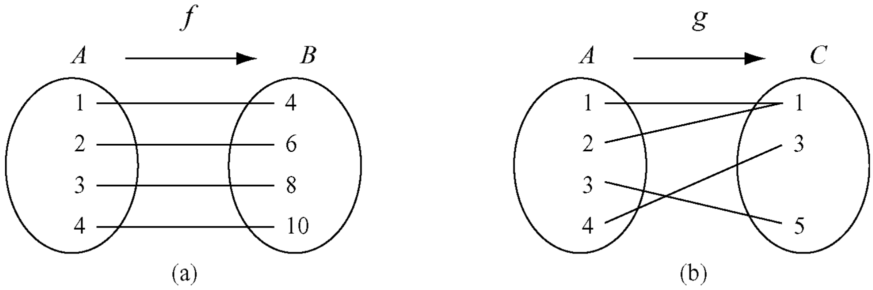

Figure 1

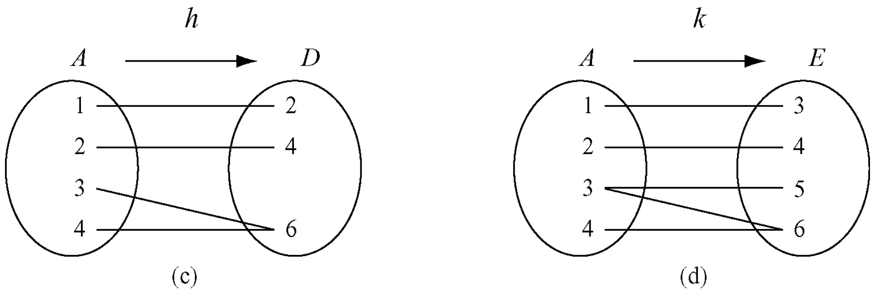

Figure 2

[Solution]

In (a), f is obviously a well-defined function. In (b), although we have $g(1)=g(2)=1$, every element of the domain A still has a unique image and so g is a function.

But the graphs in Figure 2 do not represent functions. In (c), the element 4 of the domain A does not have any image, i.e. $h(4)$ is not defined. In (d), the element 3 of the domain A of k has two different images, 5 and 6, i.e. $k(3)=5$ and $k(3)=6$. Hence, according to the definition, both h and k are not functions.

(b) Graphs of some well known functions

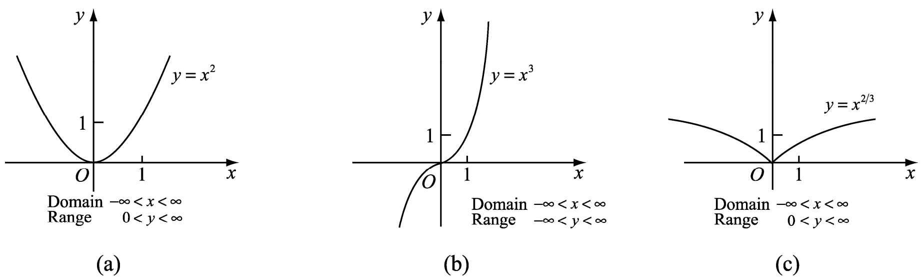

Figure 3

(c) Surjective, injective and bijective functions

(i) If $f:A\to B$ and $f(A)=B$, then f is said to be **onto** (or **surjective**). Thus f is onto if for each $y\in B$ there is at least one $x\in A$ such that $f(x)=y$.

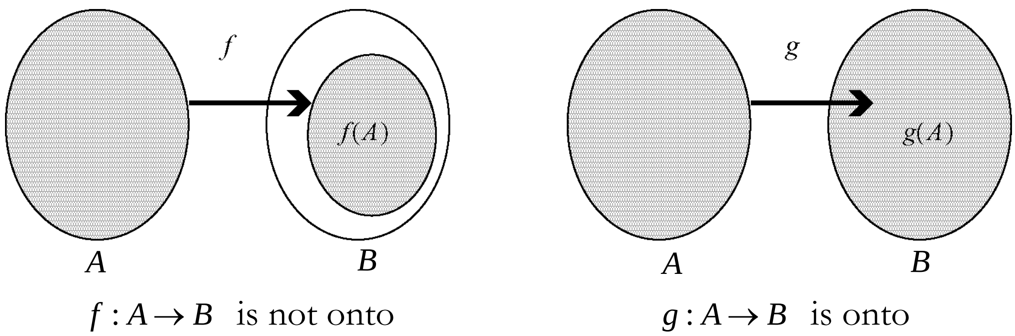

Figure 4

(ii) A function $f:A\to B$ is said to be **one-to-one** (or **injective**) if

$$f(x_1)=f(x_2) \text{ implies } x_1=x_2 \quad \forall\, x_1,x_2\in A$$

or equivalently,

$$x_1\neq x_2 \text{ implies } f(x_1)\neq f(x_2) \quad \forall\, x_1,x_2\in A.$$

That means, unequal elements in the domain have unequal images in the range. For example, the well known trigonometric function $f(x)=\sin x$, $x\in\mathbb{R}$ and n is an integer is not injective because $\sin x=\sin(x+2n\pi)$.

(iii) A mapping $f:A\to B$ is said to be **one-to-one onto** (or **bijective**) if f is both injective and surjective.

Example 2

Let $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ be a function defined by $f(z)=z-\dfrac{1}{z}$, $\forall\, z\in\mathbb{C}\setminus\{0\}$.
Prove that f is surjective.

[Solution]

Let $\omega\in\mathbb{C}$. Suppose z is pre-image of ω. Then $\omega=f(z)$,

$$\omega=z-\frac{1}{z} \quad\Rightarrow\quad z^2-\omega z-1=0$$

$$z=\frac{\omega\pm\sqrt{\omega^2+4}}{2}.$$

Now, we take $z=\dfrac{\omega\pm\sqrt{\omega^2+4}}{2}$. Next, we have to check whether the pre-image belongs to the domain $\mathbb{C}\setminus\{0\}$, and this is proved by the method of contradiction.

If $z\notin\mathbb{C}\setminus\{0\}$, then $z=0 \quad\Rightarrow\quad \dfrac{\omega\pm\sqrt{\omega^2+4}}{2}=0$

$$\omega=\pm\sqrt{\omega^2+4}$$

$$\omega^2=\omega^2+4$$

$0=4$, which is impossible.

Hence, $z\in\mathbb{C}\setminus\{0\}$, and so f is surjective.

Example 3

Let $f:\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\to[-1,1]$ defined by $f(x)=\sin x$. Is f injective?

[Solution]

Let $x_1,x_2\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. It is injective if $f(x_1)=f(x_2)$, then $x_1=x_2$.

Now, $\sin x_1-\sin x_2=2\sin\dfrac{1}{2}(x_1-x_2)\cos\dfrac{1}{2}(x_1+x_2)=0$

$\Rightarrow\quad x_1=x_2$ and so f is injective.

Example 4

Let $f:\mathbb{R}\setminus\{-1\}\to\mathbb{R}\setminus\{1\}$ be a function defined by $f(x)=\dfrac{x-1}{x+1}$ $\forall x\in\mathbb{R}\setminus\{-1\}$.

Prove that f is bijective.

[Solution]

First, we prove that f is injective, $\forall\ x_1,x_2\in\mathbb{R}\setminus\{-1\}$,

$$
\begin{aligned}
f(x_1)=f(x_2)\ &\Rightarrow\ \frac{x_1-1}{x_1+1}=\frac{x_2-1}{x_2+1}\\
&\Rightarrow\ (x_1-1)(x_2+1)=(x_2-1)(x_1+1)\\
&\Rightarrow\ x_1x_2-x_2+x_1-1=x_1x_2-x_1+x_2-1\\
&\Rightarrow\ 2x_1=2x_2\\
&\Rightarrow\ x_1=x_2\,.
\end{aligned}
$$

Hence, f is injective.

Second, we prove that f is surjective. $\forall y\in\mathbb{R}\setminus\{1\}$, we try to find the pre-image of y. By putting $y=\dfrac{x-1}{x+1}$, we have

$$
\begin{aligned}
&y(x+1)=x-1\\
&yx+y=x-1\\
&yx-x=-y-1\\
&x(y-1)=-y-1\\
&\quad x=\frac{-y-1}{y-1}\cdots\cdots(*)
\end{aligned}
$$

Since $y\in\mathbb{R}\setminus\{1\}$, $y\neq 1$ and so x exists.

To complete the proof, we need to show that $x\in\mathbb{R}\setminus\{-1\}$, i.e. $x\neq 1$.

From $(*)$, we have

$$x=\frac{-y-1}{y-1}=-1-\frac{2}{y-1}\neq -1\,.$$

Thus, f is bijective.

Note: (i) Identity mapping on any set is bijective.

(ii) A mapping of a finite set to itself is surjective iff it is injective. In contrast, there are mappings of any infinite set to itself that are injective but not surjective, and also mappings that are surjective but not injective.

Example 5

Define mapping f and g from the set of natural numbers, $\{1,2,3,\cdots\}$ to itself,

by $f(n)=2n$ and $g(n)=\begin{cases} \dfrac{(n+1)}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$

Figure 5

What properties do f and g possess?

[Solution]

f is injective but not surjective, and g is surjective but not injective. The existence of such functions is precisely what distinguishes infinite sets from finite sets. In fact, a set S can be defined as infinite if there exists a function from S to S that is injective but not surjective.

2. Graphs of Functions

A function is also described by a set of ordered pairs such that in each pair the first element belongs to the domain of the function and the second element is the corresponding element of the range; that is,

$\{(x, f(x)) : (x, f(x)) \in \mathbb{R}^2, x \in Dom(f)\}$, where $Dom(f)$ is the domain of f.

Here, each ordered pair is interpreted as the Cartesian coordinates of a point in the xy-plane. The set of all such points for a given function is the **graph** of that function. Since to each element of the domain of a function there corresponds exactly one element of the range, no vertical line can intersect the graph of a function in more than one point. Consider the following graphs.

(i) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\sqrt{16-x^2}$ has the graph of semicircle shown in Figure 6.

$$Dom(f)=[-4,4],\quad Ran(f)=[0,4]$$

where *Ran*(*f*) is the range of *f*.

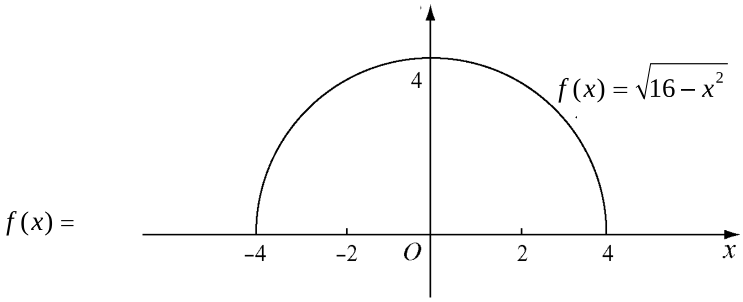

Figure 6

(ii) The set $R=\{(x,y):y^2=x, x>0\}$ is not the graph of a function for every $x>0$, as a vertical line passing through positive x values intersects the graph at two points. (see Figure 7).

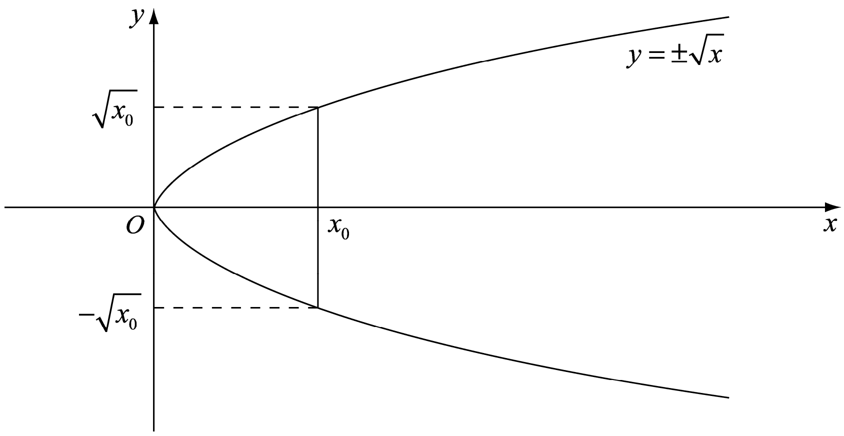

Figure 7

However, $R_1=\{(x,y):y=\sqrt{x}, x>0\}$ is the graph of a function (upper part of the parabola). So is $R_2=\{(x,y):y=-\sqrt{x}, x>0\}$.

This topic will be discussed in depth in Chapter 4 in regard to the topic of curve sketching .

3. Operations on Functions

Many functions arise as combinations of other functions. One can define the sum, difference, product and quotient of two functions to form other functions.

If f and g are functions with domains $Dom(f)$ and $Dom(g)$, then their **sum**, denoted by $f+g$, their **difference**, $f-g$, their **product**, fg and their **quotient**, f/g, are the functions defined by

$$(f+g)(x)=f(x)+g(x),$$
$$(f-g)(x)=f(x)-g(x),$$
$$(fg)(x)=f(x)g(x),$$
$$(f/g)(x)=f(x)/g(x)$$

respectively. In each case, the domain consists of all values of x belong to both $Dom(f)$ and $Dom(g)$, that is,

$$\begin{aligned}Dom(f+g)&=Dom(f-g)=Dom(fg)\\&=Dom(f/g)=Dom(f)\cap Dom(g),\end{aligned}$$

except that the values of x for which $g(x)=0$ are excluded from the domain $Dom(f/g)$.

Example 6

If f and g are functions defined by $f(x)=\dfrac{1}{x-3}$ and $g(x)=\sqrt{x}$, find $f+g$, fg, f/g, g/f and state the respective domain.

[Solution]

Function	Domain
$(f+g)(x)=\dfrac{1}{x-3}+\sqrt{x}$	$[0,\infty)\setminus\{3\}$ or $[0,3)\cup(3,\infty)$
$(fg)(x)=\dfrac{\sqrt{x}}{x-3}$	same as above
$(f/g)(x)=\dfrac{1}{\sqrt{x}(x-3)}$	$(0,3)\cup(3,\infty)$
$(g/f)(x)=\sqrt{x}(x-3)$	$[0,\infty)$

4. Composition of Functions

Suppose that the outputs of a function g can be used as inputs of a function f. We can then hook g and f together to from a new function whose inputs are the inputs of g and whose outputs are the numbers $f(g(x))$ (pronounced "f of g of x") is the composite of g and f. It is made by composing g and f in the order first g, then f. The usual stand-alone notation for this composite is $f \circ g$, which is read "f of g". Thus, the value of $f \circ g$ at x is $(f \circ g)(x) = f(g(x))$.

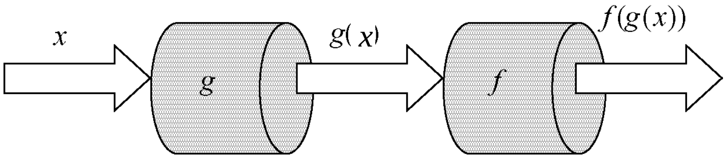

Figure 8

Example 7

Let $f(x) = \sqrt{x}$, $g(x) = -(x^2+1)$. Briefly discuss the composite functions.

[Solution]

The composite function $(f \circ g)(x) = f(g(x))$

$$= f(-(x^2+1))$$
$$= \sqrt{-(x^2+1)}\,.$$

$Dom(g) = \mathbb{R}$, so the domain of $f \circ g$ consists of all numbers x in $\mathbb{R}$ for which $-(x^2+1)$ is in the domain of f, $Dom(f) = [0,\infty)$. Therefore, $\mathrm{Dom}(f \circ g) = \phi$ and the function $f \circ g$ is not defined at any point $x \in \mathbb{R}$.

The composite function $(g \circ f)(x) = g(f(x))$

$$= g(\sqrt{x}) = -\left[(\sqrt{x})^2 + 1\right]$$
$$= -(x+1)\,.$$

$Dom(f) = [0,\infty)$, so the domain of $g \circ f$ consists of all numbers x in $[0,\infty)$ for which $\sqrt{x}$ is in the domain of g, i.e. $\sqrt{x} \in Dom(g) = \mathbb{R}$, Therefore, $Dom(g \circ f) = [0,\infty)$.

Example 8

Consider the functions $f(x)=\sqrt{x-1}$ and $g(x)=\frac{1}{x}$. Determine the composite functions $g\circ f$ and $f\circ g$ and then find $(g\circ f)(5)$ and $(f\circ g)\left(\frac{1}{4}\right)$.

[Solution]

The composite function $g\circ f$ is

$$
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g(\sqrt{x-1}) \\
&= \frac{1}{\sqrt{x-1}}.
\end{aligned}
$$

$Dom(f)=[1,\infty)$, so the domain of $g\circ f$ consists of all number x in $[1,\infty)$ for which $f(x)=\sqrt{x-1}$ is in the domain of g, that is for which $x-1\neq 0$.
Therefore

$$Dom(g\circ f)=(1,\infty).$$

The composite function $f\circ g$

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f\left(\frac{1}{x}\right)=\sqrt{\frac{1}{x}-1}.
\end{aligned}
$$

$Dom(g)=\{x:x\in\mathbb{R},x\neq 0\}$, so the domain of $f\circ g$ is the set consists of nonzero numbers x such that $\frac{1}{x}$ is in $Dom(f)=[1,\infty)$.
Therefore we have

$$Dom(f\circ g)=(0,1].$$

Finally,

$$(g\circ f)(5)=g(f(5))=\frac{1}{\sqrt{5-1}}=\frac{1}{2},$$

$$(f\circ g)\left(\frac{1}{4}\right)=f\left(g\left(\frac{1}{4}\right)\right)=\sqrt{4-1}=\sqrt{3}.$$

5. Inverses of Functions

The function defined by reversing a one-to-one function f is called the **inverse** of f. The symbol for the inverse is $f^{-1}(x)$, read "f inverse." Note that $f^{-1}(x)$ does not mean $1/f(x)$.

The result of composing f and f^{-1} in either order is the **identity function** , the function that assigns each number to itself, i.e.

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y.$$

This gives us a way to test whether two functions f and g are inverses of each other.

Notice that the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f. The following graphs illustrate how the inverse of a function related to the graph of the function.

(a) To find the value of f at x, we start at x and go up the curve and over to the y-axis.

(b) The graph of f is also the graph of f^{-1}. To find the x that gives y, we start at y and go over to the curve and down to the x-axis.

(c) To draw the graph of f^{-1}, we reflect it in the line $y = x$.

(d) Then interchange the letters x and y. Now we have a normal-looking graph of f^{-1} as a function of x.

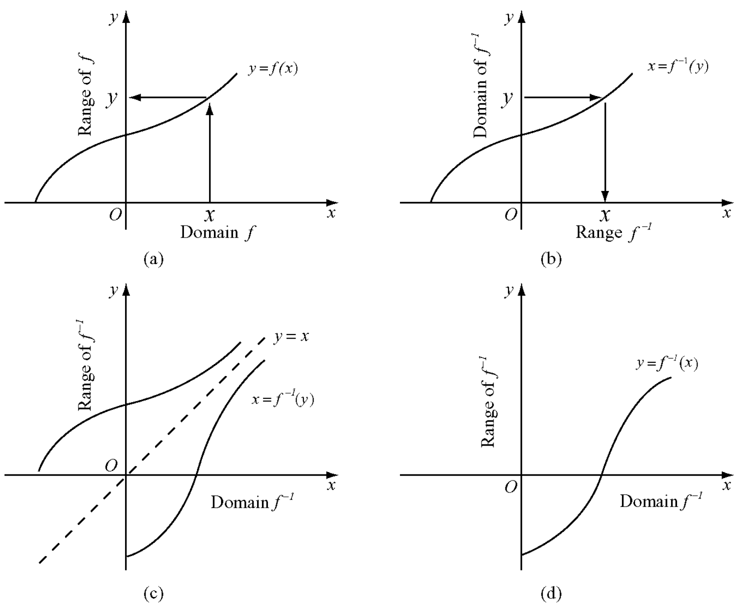

Figure 9

To express f^{-1} as a function of x, we carry out the steps:

(i) Solve the equation $y = f(x)$ for x in terms of y.

(ii) Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

Example 9

Find the inverse of $f(x) = \frac{1}{2}x + 1$, expressed as a function of x.

[Solution]

Step1: Solve $y = f(x)$ for x in terms of y:

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

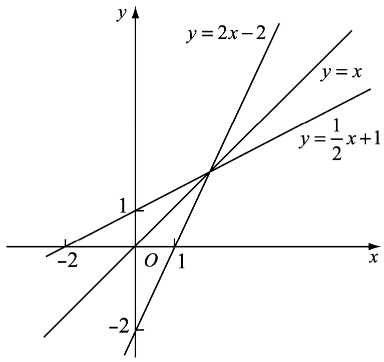

Figure 10

Step2: Interchange x and y: $y = 2x - 2$.

The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function

$$f^{-1}(x) = 2x - 2.$$

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

Also observe that the graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 10

Find the inverse of the function $y = x^2$, $x \geq 0$ expressed as a function of x.

[Solution]

Step1: Solve for x in terms of y:

$y = x^2$

$x = \sqrt{y}$. $(\because x \geq 0, \sqrt{x^2} = x)$

Step2: Interchange x and y to get $y = \sqrt{x}$.

The inverse of the function $y = x^2, x \geq 0$ is the function $y = \sqrt{x}$.

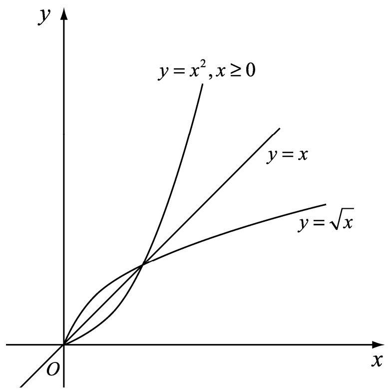

Figure 11

Notice that, unlike the restricted function $y = x^2, x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse.

6. Some special real functions

(a) Periodic Functions

The concept of periodicity is essential in describing various physical phenomena. For instance, periodic motions such as amplitude vibrations and oscillations, orbital motions of artificial satellites are encountered. A function $f(x)$ is said to be periodic with period T if it is expressed by the equation

$$f(x+T) = f(x), \quad \forall x \in \mathbb{R}.$$

For example, the sine and cosine functions are well known periodic functions; they have period 2π.

By definition, a periodic function f satisfies

$$f(x-T) = f(x-T+T) = f(x).$$

Similarly we obtain

$$f(x+2T) = f(x+T+T) = f(x+T) = f(x).$$

Hence, we can establish easily

$$f(x+nT) = f(x),$$

where n is an integer and T is the smallest positive period.

Example 11

If T is the smallest period of two functions, is it necessary for their sum and product with the same smallest period?

[Solution]

The answer is "No". For example, $f_1(x)=3\sin x+2$ & $f_2(x)=2-3\sin x$, and $g_1(x)=1+\sin x$ & $g_2(x)=1-\sin x$ have the same smallest period 2π.

On the other hand, $f_1(x)+f_2(x)=4$ and $g_1(x)\cdot g_2(x)=\cos^2 x=\dfrac{1-\cos 2x}{2}$.

The former does not have the smallest period and the latter has the smallest period π.

Example 12

(a) Let $y=f(x)$ be a periodic function with the period T. Show that the function $y=f(\omega x)$ is also a periodic function of period $\dfrac{T}{\omega}$.

(b) Find the period of the function $y=\sin(\omega x+\theta)$.

[Solution]

(a) Since $y=f(x)$ is a periodic function with the period T, we have

$$f(x)=f(x+T).$$

Hence, $f(\omega x)=f(\omega x+T)=f\left(\omega\left(x+\dfrac{T}{\omega}\right)\right)$.

Thus, the function $y=f(\omega x)$ is also a periodic function of period $\dfrac{T}{\omega}$.

(b) Let $f(x)=\sin(x+\theta)$, then

$$f(x+2\pi)=\sin(x+2\pi+\theta)=\sin(x+\theta)=f(x).$$

Thus, the period of $f(x)$ is 2π. Hence, by part (a), the period of the function $y=\sin(\omega x+\theta)=f(\omega x)$ is $\dfrac{2\pi}{\omega}$.

Example 13

Show that $f(x)=\sin x^2$ is not a periodic function.

[Solution]

We prove it by the method of contradiction. Suppose there exists a positive number ω unrelated with variable x such that

$$\sin x^2 = \sin(x+\omega)^2 .$$

When $x=0$, $\omega^2 = n\pi$.

Hence, $\sin x^2 = \sin(x+\sqrt{n\pi})^2 = \sin(x^2 + 2x\sqrt{n\pi} + n\pi)$.

If we choose $x \neq \dfrac{2k\pi - n\pi}{2\sqrt{n\pi}}$, $k \in \mathbb{R}^+$, then $2x\sqrt{n\pi} + n\pi \neq 2k\pi$.

It leads to contradiction, $\sin x^2 \neq \sin(x^2 + 2x\sqrt{n\pi} + n\pi)$.

(b) Even Functions and Odd Functions

A function $y=f(x)$ is an **even function** of x if $f(-x)=f(x)$ for every x in the function's domain. It is an **odd function** of x if $f(-x)=-f(x)$ for every x in the function's domain.

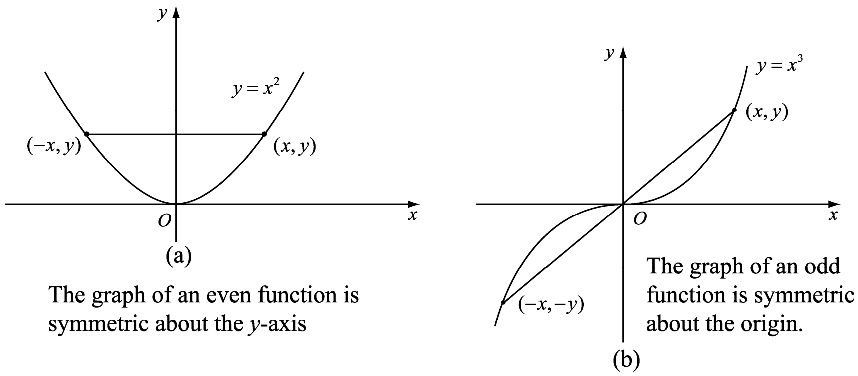

Figure 12

Consider the following:

(i)	$f(x)=x^2$ is an even function. $(-x)^2 = x^2$ for all x. The graph is symmetric about the y-axis.
(ii)	$f(x)=x$ is an odd function. $(-x)=-(x)$ for all x. The graph is symmetric about the origin.
(iii)	$f(x)=x+1$ is not odd as $f(-x)=-x+1$, which does not equal $-f(x)=-x-1$. It is not even, $\because (-x)+1 \neq x+1$.

(c) Functions defined piecewise

While some functions are defined by single formula, others are defined by applying different formulas to different parts of their domains.

For example, to graph the function shown in Figure 13, we apply different formulas to different parts of its domain.

$$y = f(x) = \begin{cases} -x \text{ if } x < 0 \\ x^2 \text{ if } 0 \le x \le 1 \\ 1 \text{ if } x > 1. \end{cases}$$

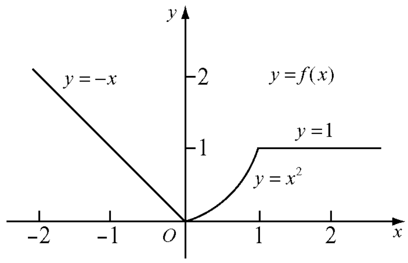

Figure 13

(d) Monotonic Functions

(i) A function is said to be **monotonically increasing** on an interval I if and only if

$$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2),\ x_1, x_2 \in I,$$

or **strictly increasing** if and only if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ x_1, x_2 \in I.$$

(ii) A function is said to be **monotonically decreasing** on an interval I if and only if

$$x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2),\ x_1, x_2 \in I.$$

or **strictly decreasing** if and only if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2),\ x_1, x_2 \in I..$$

(e) Bounded Functions

A function is said to be bounded on I if there exists a positive number M, the inequality $|f(x)| \le M$, $\forall x \in I$.

In particular,

(i) if $f(x) \le M$, f is **bounded from above** and M is called an **upper bound**.

(ii) if $f(x) \ge M$, f is **bounded from below** and M is called an **lower bound**.

Example 14

Prove that $f(x)=x^3+x$ is strictly increasing on $(-\infty,\infty)$.

[Solution]

Let $x_1<x_2$, we have

$$f(x_2)-f(x_1)=(x_2-x_1)(x_1^{\ 2}+x_1x_2+x_2^{\ 2}+1)$$

$$=(x_2-x_1)\left[\left(x_1+\frac{x_2}{2}\right)^2+\frac{3}{4}x_2^{\ 2}+1\right]>0.$$

Thus $f(x_2)>f(x_1)$ and so $f(x)$ is strictly increasing on $(-\infty,\infty)$.

Example 15

Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$.

(a) Show that

(i) $f(0)=0$,

(ii) $f(-x)=-f(x)$ for all $x\in\mathbb{R}$.

(iii) $f(nx)=nf(x)$ for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$.

(b) Show that if there exists $k>0$ such that $f(x)<k$ for all $x\in\mathbb{R}$, then $f(x)=0$ for all $x\in\mathbb{R}$.

(c) Suppose there exists $k>0$ such that $f(x)<k$ for all $x\in[0,1]$. Let $g(x)=f(x)-f(1)x$ for all $x\in\mathbb{R}$. Show that for all $x,y\in\mathbb{R}$,

(i) $g(x+y)=g(x)+g(y)$,

(ii) $g(x+1)=g(x)$,

(iii) $f(x)<k+|f(1)|$.

Hence or otherwise, show that $f(x)=f(1)x$ for all $x\in\mathbb{R}$.

[Solution]

(a) (i) $f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$.

(ii) $f(x+(-x))=f(x)+f(-x)$

$f(0)=f(x)+f(-x)$

$f(-x)=-f(x)$.

(iii) Let n be a non-negative integer.

The result is to be proved by induction on n.

First, when $n=0$, from (a) (i),

$$f(0\cdot x)=f(0)=0=0\cdot f(x).$$

Hence, the proposition is true for $n=0$.

Second, assume $f(kx)=kf(x)$, where k is a non-negattive integer.

Then $f((k+1)x)=f(kx+x)=kf(x)+f(x)=(k+1)f(x)$.

Hence, the proposition is also true for $n=k+1$.

By the principle of mathematical induction, the proposition is true for all non-negative integer n.

The case of n being a negative integer is proved as follows.

$$f(nx)=f(-(-nx))=-(-nf(x))=nf(x).$$

(b) The result is to be proved by method of contradiction. Suppose there exists $x_0\in\mathbb{R}$ such that $f(x_0)\neq 0$.

Case 1: $f(x_0)>0$,

For all $k>0$, there exists a positive integer n such that $n\geq\dfrac{k}{f(x_0)}$.

Since $f(nx_0)=nf(x_0)\geq\dfrac{k}{f(x_0)}f(x_0)=k$, which is a contradiction.

Case 2: $f(x_0)<0$,

For all $k>0$, there exists a positive integer n such that $n\leq\dfrac{k}{f(x_0)}$.

Since $f(nx_0)=nf(x_0)\geq\dfrac{k}{f(x_0)}f(x_0)=k$ $\left[\because f(x_0)<0\right]$, it is a contradiction. Therefore, $f(x)=0\quad\forall x\in\mathbb{R}$.

(c) (i)
$$\begin{aligned} g(x+y)&=f(x+y)-f(1)(x+y)\\ &=f(x)+f(y)-f(1)x-f(1)y\\ &=[f(x)-f(1)x]+[f(y)-f(1)y]\\ &=g(x)+g(y). \end{aligned}$$

(ii) $g(x+1)=g(x)+g(1)=g(x)+[f(1)-f(1)1]=g(x)$.

(iii) Let $[x]$ be the greatest integer $\leq x$.

Then $x=[x]+y$ where $0\leq y\leq 1$.

$\because$ From (c) (i), $g(y)=g(y+1)=\cdots=g(y+[x])$

$\therefore$ $g(x)=g(y)$.

Hence, $g(x) = g(y)$

$$\begin{aligned}
&= f(y) - f(1)y \\
&< k - f(1)y \\
&< \left|k - f(1)y\right| \\
&< k + |f(1)|\, y \quad \left[\because\ y \in [0,1)\right] \\
&< k + |f(1)|. \quad \left[\because\ 0 \le y < 1\right]
\end{aligned}$$

From (c) (i) and (iii), g satisfies the condition of f stated in (b).

Hence,

$$g(x) = 0 \quad \forall x \in \mathbb{R}.$$

$$f(x) - f(1)x = 0$$

$$f(x) = f(1)x.$$