Rational Expressions

We now need to look at rational expressions. A **rational expression** is nothing more than a fraction in which the numerator and/or the denominator are polynomials. Here are some examples of rational expressions.

$$\frac{6}{x-1} \qquad \frac{z^2-1}{z^2+5} \qquad \frac{m^4+18m+1}{m^2-m-6} \qquad \frac{4x^2+6x-10}{1}$$

The last one may look a little strange since it is more commonly written $4x^2+6x-10$. However, it's important to note that polynomials can be thought of as rational expressions if we need to, although they rarely are.

There is an unspoken rule when dealing with rational expressions that we now need to address. When dealing with numbers we know that division by zero is not allowed. Well the same is true for rational expressions. So, when dealing with rational expressions we will always assume that whatever x is it won't give division by zero. We rarely write these restrictions down, but we will always need to keep them in mind.

For the first one listed we need to avoid $x=1$. The second rational expression is never zero in the denominator and so we don't need to worry about any restrictions. Note as well that the numerator of the second rational expression will be zero. That is okay, we just need to avoid division by zero. For the third rational expression we will need to avoid $m=3$ and $m=-2$. The final rational expression listed above will never be zero in the denominator so again we don't need to have any restrictions.

The first topic that we need to discuss here is reducing a rational expression to lowest terms. A rational expression has been **reduced to lowest terms** if all common factors from the numerator and denominator have been canceled. We already know how to do this with number fractions so let's take a quick look at an example.

$$\text{not reduced to lowest terms } \Rightarrow \; \frac{12}{8}=\frac{\cancel{(4)}(3)}{\cancel{(4)}(2)}=\frac{3}{2} \; \Leftarrow \text{ reduced to lowest terms}$$

With rational expression it works exactly the same way.

$$\text{not reduced to lowest terms } \Rightarrow \; \frac{\cancel{(x+3)}(x-1)}{x\cancel{(x+3)}}=\frac{x-1}{x} \; \Leftarrow \text{ reduced to lowest terms}$$

We do have to be careful with canceling however. There are some common mistakes that students often make with these problems. Recall that in order to cancel a factor it must multiply the whole numerator and the whole denominator. So, the $x+3$ above could cancel since it multiplied the whole numerator and the whole denominator. However, the x's in the reduced form can't cancel since the x in the numerator is not times the whole numerator.

To see why the x's don't cancel in the reduced form above put a number in and see what happens. Let's plug in $x=4$.

$$\frac{4-1}{4}=\frac{3}{4} \qquad\qquad \frac{\not{4}-1}{\not{4}}=-1$$

Clearly the two aren't the same number!

So, be careful with canceling. As a general rule of thumb remember that you can't cancel something if it's got a "+" or a "-" on one side of it. There is one exception to this rule of thumb with "-" that we'll deal with in an example later on down the road.

Let's take a look at a couple of examples.

Example 1 Reduce the following rational expression to lowest terms.

(a) $\dfrac{x^2-2x-8}{x^2-9x+20}$ [Solution]

(b) $\dfrac{x^2-25}{5x-x^2}$ [Solution]

(c) $\dfrac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}$ [Solution]

Solution

When reducing a rational expression to lowest terms the first thing that we will do is factor both the numerator and denominator as much as possible. That should always be the first step in these problems.

Also, the factoring in this section, and all successive section for that matter, will be done without explanation. It will be assumed that you are capable of doing and/or checking the factoring on your own. In other words, make sure that you can factor!

(a) $\dfrac{x^2-2x-8}{x^2-9x+20}$

We'll first factor things out as completely as possible. Remember that we can't cancel anything at this point in time since every term has a "+" or a "-" on one side of it! We've got to factor first!

$$\frac{x^2-2x-8}{x^2-9x+20}=\frac{\left(x-4\right)\left(x+2\right)}{\left(x-5\right)\left(x-4\right)}$$

At this point we can see that we've got a common factor in both the numerator and the denominator and so we can cancel the x-4 from both. Doing this gives,

$$\frac{x^2-2x-8}{x^2-9x+20}=\frac{x+2}{x-5}$$

This is also all the farther that we can go. Nothing else will cancel and so we have reduced this expression to lowest terms.

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(b) $\dfrac{x^2-25}{5x-x^2}$

Again, the first thing that we'll do here is factor the numerator and denominator.

$$\frac{x^2-25}{5x-x^2}=\frac{(x-5)(x+5)}{x(5-x)}$$

At first glance it looks there is nothing that will cancel. Notice however that there is a term in the denominator that is almost the same as a term in the numerator except all the signs are the opposite.

We can use the following fact on the second term in the denominator.

$$a-b=-(b-a) \qquad \text{OR} \qquad -a+b=-(a-b)$$

This is commonly referred to as **factoring a minus sign out** because that is exactly what we've done. There are two forms here that cover both possibilities that we are liable to run into. In our case however we need the first form.

Because of some notation issues let's just work with the denominator for a while.

$$\begin{aligned} x(5-x) &= x\left[-(x-5)\right] \\ &= x\left[(-1)(x-5)\right] \\ &= x(-1)(x-5) \\ &= (-1)(x)(x-5) \\ &= -x(x-5) \end{aligned}$$

Notice the steps used here. In the first step we factored out the minus sign, but we are still multiplying the terms and so we put in an added set of brackets to make sure that we didn't forget that. In the second step we acknowledged that a minus sign in front is the same as multiplication by "-1". Once we did that we didn't really need the extra set of brackets anymore so we dropped them in the third step. Next, we recalled that we change the order of a multiplication if we need to so we flipped the x and the "-1". Finally, we dropped the "-1" and just went back to a negative sign in the front.

Typically when we factor out minus signs we skip all the intermediate steps and go straight to the final step.

Let's now get back to the problem. The rational expression becomes,

$$\frac{x^2-25}{5x-x^2}=\frac{(x-5)(x+5)}{-x(x-5)}$$

At this point we can see that we do have a common factor and so we can cancel the $x-5$.

$$\frac{x^2-25}{5x-x^2}=\frac{x+5}{-x}=-\frac{x+5}{x}$$

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(c) $\dfrac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}$

In this case the denominator is already factored for us to make our life easier. All we need to do is factor the numerator.

$$\frac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}=\frac{x^5\left(x^2+2x+1\right)}{x^3\left(x+1\right)^8}=\frac{x^5\left(x+1\right)^2}{x^3\left(x+1\right)^8}$$

Now we reach the point of this part of the example. There are 5 x's in the numerator and 3 in the denominator so when we cancel there will be 2 left in the numerator. Likewise, there are 2 $\left(x+1\right)$'s in the numerator and 8 in the denominator so when we cancel there will be 6 left in the denominator. Here is the rational expression reduced to lowest terms.

$$\frac{x^7+2x^6+x^5}{x^3\left(x+1\right)^8}=\frac{x^2}{\left(x+1\right)^6}$$

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Before moving on let's briefly discuss the answer in the second part of this example. Notice that we moved the minus sign from the denominator to the front of the rational expression in the final form. This can always be done when we need to. Recall that the following are all equivalent.

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

In other words, a minus sign in front of a rational expression can be moved onto the whole numerator or whole denominator if it is convenient to do that. We do have to be careful with this however. Consider the following rational expression.

$$\frac{-x+3}{x+1}$$

In this case the "-" on the x can't be moved to the front of the rational expression since it is only on the x. In order to move a minus sign to the front of a rational expression it needs to be times the whole numerator or denominator. So, if we factor a minus out of the numerator we could then move it into the front of the rational expression as follows,

$$\frac{-x+3}{x+1}=\frac{-\left(x-3\right)}{x+1}=-\frac{x-3}{x+1}$$

The moral here is that we need to be careful with moving minus signs around in rational expressions.

We now need to move into adding, subtracting, multiplying and dividing rational expressions.

Let's start with multiplying and dividing rational expressions. The general formulas are as follows,

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Note the two different forms for denoting division. We will use either as needed so make sure you are familiar with both. Note as well that to do division of rational expressions all that we need to do is multiply the numerator by the reciprocal of the denominator (*i.e.* the fraction with the numerator and denominator switched).

Before doing a couple of examples there are a couple of *special* cases of division that we should look at. In the general case above both the numerator and the denominator of the rational expression where fractions, however, what if one of them isn't a fraction. So let's look at the following cases.

$$\frac{a}{\frac{c}{d}} \qquad \qquad \frac{\frac{a}{b}}{c}$$

Students often make mistakes with these initially. To correctly deal with these we will turn the numerator (first case) or denominator (second case) into a fraction and then do the general division on them.

$$\frac{a}{\frac{c}{d}} = \frac{\frac{a}{1}}{\frac{c}{d}} = \frac{a}{1} \cdot \frac{d}{c} = \frac{ad}{c}$$

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$$

Be careful with these cases. It is easy to make a mistake with these and incorrectly do the division.

Now let's take a look at a couple of examples.

Example 2 Perform the indicated operation and reduce the answer to lowest terms.

(a) $\dfrac{x^2-5x-14}{x^2-3x+2} \cdot \dfrac{x^2-4}{x^2-14x+49}$ [Solution]

(b) $\dfrac{m^2-9}{m^2+5m+6} \div \dfrac{3-m}{m+2}$ [Solution]

(c) $\dfrac{y^2+5y+4}{\dfrac{y^2-1}{y+5}}$ [Solution]

Solution

Notice that with this problem we have started to move away from x as the main variable in the examples. Do not get so used to seeing x's that you always expect them. The problems will work the same way regardless of the letter we use for the variable so don't get excited about the different letters here.

(a) $\dfrac{x^2-5x-14}{x^2-3x+2}\cdot\dfrac{x^2-4}{x^2-14x+49}$

Okay, this is a multiplication. The first thing that we should always do in the multiplication is to factor everything in sight as much as possible.

$$\frac{x^2-5x-14}{x^2-3x+2}\cdot\frac{x^2-4}{x^2-14x+49}=\frac{(x-7)(x+2)}{(x-2)(x-1)}\cdot\frac{(x-2)(x+2)}{(x-7)^2}$$

Now, recall that we can cancel things across a multiplication as follows.

$$\frac{a}{b\not{k}}\cdot\frac{c\not{k}}{d}=\frac{a}{b}\cdot\frac{c}{d}$$

Note that this ONLY works for multiplication and NOT for division!

In this case we do have multiplication so cancel as much as we can and then do the multiplication to get the answer.

$$\frac{x^2-5x-14}{x^2-3x+2}\cdot\frac{x^2-4}{x^2-14x+49}=\frac{(x+2)}{(x-1)}\cdot\frac{(x+2)}{(x-7)}=\frac{(x+2)^2}{(x-1)(x-7)}$$

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(b) $\dfrac{m^2-9}{m^2+5m+6}\div\dfrac{3-m}{m+2}$

With division problems it is very easy to mistakenly cancel something that shouldn't be canceled and so the first thing we do here (before factoring!!!!) is do the division. Once we've done the division we have a multiplication problem and we factor as much as possible, cancel everything that can be canceled and finally do the multiplication.

So, let's get started on this problem.

$$\begin{aligned}\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}&=\frac{m^2-9}{m^2+5m+6}\cdot\frac{m+2}{3-m}\\&=\frac{(m-3)(m+3)}{(m+3)(m+2)}\cdot\frac{(m+2)}{(3-m)}\end{aligned}$$

Now, notice that there will be a lot of canceling here. Also notice that if we factor a minus sign out of the denominator of the second rational expression. Let's do some of the canceling and then do the multiplication.

$$\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}=\frac{(m-3)}{1}\cdot\frac{1}{-(m-3)}=\frac{(m-3)}{-(m-3)}$$

Remember that when we cancel all the terms out of a numerator or denominator there is actually a "1" left over! Now, we didn't finish the canceling to make a point. Recall that at the start of this discussion we said that as a rule of thumb we can only cancel terms if there isn't a "+" or a "-" on either side of it with one exception for the "-". We are now at that exception. If there is a "-" if front of the whole numerator or denominator, as we've got here, then we can still cancel the term. In this case the "-" acts as a "-1" that is multiplied by the whole denominator and so is a factor instead of an addition or subtraction. Here is the final answer for this part.

$$\frac{m^2-9}{m^2+5m+6} \div \frac{3-m}{m+2} = \frac{1}{-1} = -1$$

In this case all the terms canceled out and we were left with a number. This doesn't happen all that often, but as this example has shown it clearly can happen every once in a while so don't get excited about it when it does happen.

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(c) $\dfrac{y^2+5y+4}{\dfrac{y^2-1}{y+5}}$

This is one of the special cases for division. So, as with the previous part, we will first do the division and then we will factor and cancel as much as we can.

Here is the work for this part.

$$\frac{y^2+5y+4}{\dfrac{y^2-1}{y+5}} = \frac{\left(y^2+5y+4\right)\left(y+5\right)}{y^2-1} = \frac{\left(y+1\right)\left(y+4\right)\left(y+5\right)}{\left(y+1\right)\left(y-1\right)} = \frac{\left(y+4\right)\left(y+5\right)}{y-1}$$

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Okay, it's time to move on to addition and subtraction of rational expressions. Here are the general formulas.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad\qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

As these have shown we've got to remember that in order to add or subtract rational expression or fractions we MUST have common denominators. If we don't have common denominators then we need to first get common denominators.

Let's remember how do to do this with a quick number example.

$$\frac{5}{6} - \frac{3}{4}$$

In this case we need a common denominator and recall that it's usually best to use the **least common denominator**, often denoted **lcd**. In this case the least common denominator is 12. So we need to get the denominators of these two fractions to a 12. This is easy to do. In the first case we need to multiply the denominator by 2 to get 12 so we will multiply the numerator and denominator of the first fraction by 2. Remember that we've got to multiply both the numerator and denominator by the same number since we aren't allowed to actually change the problem and this is equivalent to multiplying the fraction by 1 since $\frac{a}{a}=1$. For the second term we'll need to multiply the numerator and denominator by a 3.

$$\frac{5}{6}-\frac{3}{4}=\frac{5(2)}{6(2)}-\frac{3(3)}{4(3)}=\frac{10}{12}-\frac{9}{12}=\frac{10-9}{12}=\frac{1}{12}$$

Now, the process for rational expressions is identical. The main difficulty is in finding the least common denominator. However, there is a really simple process for finding the least common denominator for rational expressions. Here is it.

1. Factor all the denominators.
2. Write down each factor that appears at least once in any of the denominators. Do NOT write down the power that is on each factor, only write down the factor
3. Now, for each factor written down in the previous step write down the largest power that occurs in all the denominators containing that factor.
4. The product all the factors from the previous step is the least common denominator.

Let's work some examples.

Example 3 Perform the indicated operation.

(a) $\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}$ [Solution]

(b) $\frac{2}{z+1}-\frac{z-1}{z+2}$ [Solution]

(c) $\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}$ [Solution]

(d) $\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}$ [Solution]

(e) $\frac{4}{y+2}-\frac{1}{y}+1$ [Solution]

Solution

(a) $\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}$

For this problem there are coefficients on each term in the denominator so we'll first need the least common denominator for the coefficients. This is 6. Now, x (by itself with a power of 1) is the only factor that occurs in any of the denominators. So, the least common denominator for this part is x with the largest power that occurs on all the x's in the problem, which is 5. So, the least common denominator for this set of rational expression is

$$\text{lcd} : 6x^5$$

So, we simply need to multiply each term by an appropriate quantity to get this in the denominator and then do the addition and subtraction. Let's do that.

$$\frac{4}{6x^2}-\frac{1}{3x^5}+\frac{5}{2x^3}=\frac{4\left(x^3\right)}{6x^2\left(x^3\right)}-\frac{1\left(2\right)}{3x^5\left(2\right)}+\frac{5\left(3x^2\right)}{2x^3\left(3x^2\right)}$$
$$=\frac{4x^3}{6x^5}-\frac{2}{6x^5}+\frac{15x^2}{6x^5}$$
$$=\frac{4x^3-2+15x^2}{6x^5}$$

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(b) $\dfrac{2}{z+1}-\dfrac{z-1}{z+2}$

In this case there are only two factors and they both occur to the first power and so the least common denominator is.

$$\text{lcd}:\ \left(z+1\right)\left(z+2\right)$$

Now, in determining what to multiply each part by simply compare the current denominator to the least common denominator and multiply top and bottom by whatever is "missing". In the first term we're "missing" a $z+2$ and so that's what we multiply the numerator and denominator by. In the second term we're "missing" a $z+1$ and so that's what we'll multiply in that term.

Here is the work for this problem.

$$\frac{2}{z+1}-\frac{z-1}{z+2}=\frac{2\left(z+2\right)}{\left(z+1\right)\left(z+2\right)}-\frac{\left(z-1\right)\left(z+1\right)}{\left(z+2\right)\left(z+1\right)}=\frac{2\left(z+2\right)-\left(z-1\right)\left(z+1\right)}{\left(z+1\right)\left(z+2\right)}$$

The final step is to do any multiplication in the numerator and simplify that up as much as possible.

$$\frac{2}{z+1}-\frac{z-1}{z+2}=\frac{2z+4-\left(z^2-1\right)}{\left(z+1\right)\left(z+2\right)}=\frac{2z+4-z^2+1}{\left(z+1\right)\left(z+2\right)}=\frac{-z^2+2z+5}{\left(z+1\right)\left(z+2\right)}$$

Be careful with minus signs and parenthesis when doing the subtraction.

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(c) $\dfrac{y}{y^2-2y+1}-\dfrac{2}{y-1}+\dfrac{3}{y+2}$

Let's first factor the denominators and determine the least common denominator.

$$\frac{y}{\left(y-1\right)^2}-\frac{2}{y-1}+\frac{3}{y+2}$$

So, there are two factors in the denominators a *y-1* and a *y+2*. So we will write both of those down and then take the highest power for each. That means a 2 for the *y-1* and a 1 for the *y+2*. Here is the least common denominator for this rational expression.

$$\text{lcd}:\ \left(y+2\right)\left(y-1\right)^2$$

Now determine what's missing in the denominator for each term, multiply the numerator and denominator by that and then finally do the subtraction and addition.

$$\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}=\frac{y(y+2)}{(y-1)^2(y+2)}-\frac{2(y-1)(y+2)}{(y-1)(y-1)(y+2)}+\frac{3(y-1)^2}{(y-1)^2(y+2)}$$
$$=\frac{y(y+2)-2(y-1)(y+2)+3(y-1)^2}{(y-1)^2(y+2)}$$

Okay now let's multiply the numerator out and simplify. In the last term recall that we need to do the multiplication prior to distributing the 3 through the parenthesis. Here is the simplification work for this part.

$$\frac{y}{y^2-2y+1}-\frac{2}{y-1}+\frac{3}{y+2}=\frac{y^2+2y-2(y^2+y-2)+3(y^2-2y+1)}{(y-1)^2(y+2)}$$
$$=\frac{y^2+2y-2y^2-2y+4+3y^2-6y+3}{(y-1)^2(y+2)}$$
$$=\frac{2y^2-6y+7}{(y-1)^2(y+2)}$$

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(d) $\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}$

Again, factor the denominators and get the least common denominator.

$$\frac{2x}{(x-3)(x+3)}-\frac{1}{x+3}-\frac{2}{x-3}$$

The least common denominator is,

$$\text{lcd}: (x-3)(x+3)$$

Notice that the first rational expression already contains this in its denominator, but that is okay. In fact, because of that the work will be slightly easier in this case. Here is the subtraction for this problem.

$$\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}=\frac{2x}{(x-3)(x+3)}-\frac{1(x-3)}{(x+3)(x-3)}-\frac{2(x+3)}{(x-3)(x+3)}$$
$$=\frac{2x-(x-3)-2(x+3)}{(x-3)(x+3)}$$
$$=\frac{2x-x+3-2x-6}{(x-3)(x+3)}$$
$$=\frac{-x-3}{(x-3)(x+3)}$$

Notice that we can actually go one step further here. If we factor a minus out of the numerator we can do some canceling.

$$\frac{2x}{x^2-9}-\frac{1}{x+3}-\frac{2}{x-3}=\frac{-(x+3)}{(x-3)(x+3)}=\frac{-1}{x-3}$$

Sometimes this kind of canceling will happen after the addition/subtraction so be on the lookout for it.

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(e) $\frac{4}{y+2}-\frac{1}{y}+1$

The point of this problem is that "1" sitting out behind everything. That isn't really the problem that it appears to be. Let's first rewrite things a little here.

$$\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}$$

In this way we see that we really have three fractions here. One of them simply has a denominator of one. The least common denominator for this part is,

$$\text{lcd}: y(y+2)$$

Here is the addition and subtraction for this problem.

$$\begin{aligned}\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}&=\frac{4y}{(y+2)(y)}-\frac{y+2}{y(y+2)}+\frac{y(y+2)}{y(y+2)}\\&=\frac{4y-(y+2)+y(y+2)}{y(y+2)}\end{aligned}$$

Notice the set of parenthesis we added onto the second numerator as we did the subtraction. We are subtracting off the whole numerator and so we need the parenthesis there to make sure we don't make any mistakes with the subtraction.

Here is the simplification for this rational expression.

$$\frac{4}{y+2}-\frac{1}{y}+\frac{1}{1}=\frac{4y-y-2+y^2+2y}{y(y+2)}=\frac{y^2+5y-2}{y(y+2)}$$

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