Polynomials

In this section we will start looking at polynomials. Polynomials will show up in pretty much every section of every chapter in the remainder of this material and so it is important that you understand them.

We will start off with **polynomials in one variable**. Polynomials in one variable are algebraic expressions that consist of terms in the form ax^n where n is a non-negative (*i.e.* positive or zero) integer and a is a real number and is called the **coefficient** of the term. The **degree** of a polynomial in one variable is the largest exponent in the polynomial.

Note that we will often drop the "in one variable" part and just say polynomial.

Here are examples of polynomials and their degrees.

$$5x^{12} - 2x^6 + x^5 - 198x + 1 \qquad \text{degree} : 12$$
$$x^4 - x^3 + x^2 - x + 1 \qquad \text{degree} : 4$$
$$56x^{23} \qquad \text{degree} : 23$$
$$5x - 7 \qquad \text{degree} : 1$$
$$-8 \qquad \text{degree} : 0$$

So, a polynomial doesn't have to contain all powers of x as we see in the first example. Also, polynomials can consist of a single term as we see in the third and fifth example.

We should probably discuss the final example a little more. This really is a polynomial even it may not look like one. Remember that a polynomial is any algebraic expression that consists of terms in the form ax^n. Another way to write the last example is

$$-8x^0$$

Written in this way makes it clear that the exponent on the x is a zero (this also explains the degree…) and so we can see that it really is a polynomial in one variable.

Here are some examples of things that aren't polynomials.

$$4x^6 + 15x^{-8} + 1$$
$$5\sqrt{x} - x + x^2$$
$$\frac{2}{x} + x^3 - 2$$

The first one isn't a polynomial because it has a negative exponent and all exponents in a polynomial must be positive.

To see why the second one isn't a polynomial let's rewrite it a little.

$$5\sqrt{x} - x + x^2 = 5x^{\frac{1}{2}} - x + x^2$$

By converting the root to exponent form we see that there is a rational root in the algebraic expression. All the exponents in the algebraic expression must be integers in order for the algebraic expression to be a polynomial. As a general rule of thumb if an algebraic expression has a radical in it then it isn't a polynomial.

Let's also rewrite the third one to see why it isn't a polynomial.

$$\frac{2}{x} + x^3 - 2 = 2x^{-1} + x^3 - 2$$

So, this algebraic expression really has a negative exponent in it and we know that isn't allowed. Another rule of thumb is if there are any variables in the denominator of a fraction then the algebraic expression isn't a polynomial.

Note that this doesn't mean that radicals and fractions aren't allowed in polynomials. They just can't involve the variables. For instance, the following is a polynomial

$$\sqrt[3]{5}\,x^4 - \frac{7}{12}x^2 + \frac{1}{\sqrt{8}}x - 5\sqrt[14]{113}$$

There are lots of radicals and fractions in this algebraic expression, but the denominators of the fractions are only numbers and the radicands of each radical are only a numbers. Each x in the algebraic expression appears in the numerator and the exponent is a positive (or zero) integer. Therefore this is a polynomial.

Next, let's take a quick look at **polynomials in two variables**. Polynomials in two variables are algebraic expressions consisting of terms in the form $ax^n y^m$. The degree of each term in a polynomial in two variables is the sum of the exponents in each term and the **degree** of the polynomial is the largest such sum.

Here are some examples of polynomials in two variables and their degrees.

$$x^2y - 6x^3y^{12} + 10x^2 - 7y + 1 \qquad \text{degree : } 15$$

$$6x^4 + 8y^4 - xy^2 \qquad \text{degree : } 4$$

$$x^4y^2 - x^3y^3 - xy + x^4 \qquad \text{degree : } 6$$

$$6x^{14} - 10y^3 + 3x - 11y \qquad \text{degree : } 14$$

In these kinds of polynomials not every term needs to have both x's and y's in them, in fact as we see in the last example they don't need to have any terms that contain both x's and y's. Also, the degree of the polynomial may come from terms involving only one variable. Note as well that multiple terms may have the same degree.

We can also talk about polynomials in three variables, or four variables or as many variables as we need. The vast majority of the polynomials that we'll see in this course are polynomials in one variable and so most of the examples in the remainder of this section will be polynomials in one variable.

Next we need to get some terminology out of the way. A **monomial** is a polynomial that consists of exactly one term. A **binomial** is a polynomial that consists of exactly two terms. Finally, a **trinomial** is a polynomial that consists of exactly three terms. We will use these terms off and on so you should probably be at least somewhat familiar with them.

Now we need to talk about adding, subtracting and multiplying polynomials. You'll note that we left out division of polynomials. That will be discussed in a later section where we will use division of polynomials quite often.

Before actually starting this discussion we need to recall the distributive law. This will be used repeatedly in the remainder of this section. Here is the distributive law.

$$a(b+c) = ab + ac$$

We will start with adding and subtracting polynomials. This is probably best done with a couple of examples.

Example 1 Perform the indicated operation for each of the following.

(a) Add $6x^5 - 10x^2 + x - 45$ to $13x^2 - 9x + 4$. [Solution]

(b) Subtract $5x^3 - 9x^2 + x - 3$ from $x^2 + x + 1$. [Solution]

Solution

(a) Add $6x^5 - 10x^2 + x - 45$ to $13x^2 - 9x + 4$.

The first thing that we should do is actually write down the operation that we are being asked to do.

$$\left(6x^5 - 10x^2 + x - 45\right) + \left(13x^2 - 9x + 4\right)$$

In this case the parenthesis are not required since are adding the two polynomials. They are there simply to make clear the operation that we are performing. To add two polynomials all that we do is **combine like terms**. This means that for each term with the same exponent we will add or subtract the coefficient of that term.

In this case this is,

$$\begin{aligned}\left(6x^5 - 10x^2 + x - 45\right) + \left(13x^2 - 9x + 4\right) &= 6x^5 + \left(-10 + 13\right)x^2 + \left(1 - 9\right)x - 45 + 4 \\ &= 6x^5 + 3x^2 - 8x - 41\end{aligned}$$

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(b) Subtract $5x^3 - 9x^2 + x - 3$ from $x^2 + x + 1$.

Again, let's write down the operation we are doing here. We will also need to be very careful with the order that we write things down in. Here is the operation

$$x^2 + x + 1 - \left(5x^3 - 9x^2 + x - 3\right)$$

This time the parentheses around the second term are absolutely required. We are subtracting the whole polynomial and the parenthesis must be there to make sure we are in fact subtracting the whole polynomial.

In doing the subtraction the first thing that we'll do is **distribute the minus sign** through the parenthesis. This means that we will change the sign on every term in the second polynomial. Note that all we are really doing here is multiplying a "-1" through the second polynomial using the distributive law. After distributing the minus through the parenthesis we again combine like terms.

Here is the work for this problem.

$$\begin{aligned}x^2 + x + 1 - \left(5x^3 - 9x^2 + x - 3\right) &= x^2 + x + 1 - 5x^3 + 9x^2 - x + 3 \\ &= -5x^3 + 10x^2 + 4\end{aligned}$$

Note that sometimes a term will completely drop out after combing like terms as the x did here. This will happen on occasion so don't get excited about it when it does happen.

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Now let's move onto multiplying polynomials. Again, it's best to do these in an example.

Example 2 Multiply each of the following.

(a) $4x^2\left(x^2-6x+2\right)$ [Solution]

(b) $\left(3x+5\right)\left(x-10\right)$ [Solution]

(c) $\left(4x^2-x\right)\left(6-3x\right)$ [Solution]

(d) $\left(3x+7y\right)\left(x-2y\right)$ [Solution]

(e) $\left(2x+3\right)\left(x^2-x+1\right)$ [Solution]

Solution

(a) $4x^2\left(x^2-6x+2\right)$

This one is nothing more than a quick application of the distributive law.

$$4x^2\left(x^2-6x+2\right)=4x^4-24x^3+8x^2$$

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(b)

$\left(3x+5\right)\left(x-10\right)$ This one will use the FOIL method for multiplying these two binomials.

$$\left(3x+5\right)\left(x-10\right)=\underbrace{3x^2}_{\text{First Terms}}-\underbrace{30x}_{\text{Outer Terms}}+\underbrace{5x}_{\text{Inner Terms}}-\underbrace{50}_{\text{Last Terms}}=3x^2-25x-50$$

Recall that the FOIL method will only work when multiplying two binomials. If either of the polynomials isn't a binomial then the FOIL method won't work.

Also note that all we are really doing here is multiplying every term in the second polynomial by every term in the first polynomial. The FOIL acronym is simply a convenient way to remember this.

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(c) $\left(4x^2-x\right)\left(6-3x\right)$

Again we will just FOIL this one out.

$$\left(4x^2-x\right)\left(6-3x\right)=24x^2-12x^3-6x+3x^2=-12x^3+27x^2-6x$$

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(d) $\left(3x+7y\right)\left(x-2y\right)$

We can still FOIL binomials that involve more than one variable so don't get excited about these kinds of problems when they arise.

$$\left(3x+7y\right)\left(x-2y\right)=3x^2-6xy+7xy-14y^2=3x^2+xy-14y^2$$

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(e) $\left(2x+3\right)\left(x^2-x+1\right)$

In this case the FOIL method won't work since the second polynomial isn't a binomial. Recall however that the FOIL acronym was just a way to remember that we multiply every term in the second polynomial by every term in the first polynomial.

That is all that we need to do here.

$$\left(2x+3\right)\left(x^2-x+1\right)=2x^3-2x^2+2x+3x^2-3x+3=2x^3+x^2-x+3$$

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Let's work another set of examples that will illustrate some nice formulas for some special products. We will give the formulas after the example.

Example 3 Multiply each of the following.

(a) $\left(3x+5\right)\left(3x-5\right)$ [Solution]

(b) $\left(2x+6\right)^2$ [Solution]

(c) $\left(1-7x\right)^2$ [Solution]

(d) $4\left(x+3\right)^2$ [Solution]

Solution

(a) $\left(3x+5\right)\left(3x-5\right)$

We can use FOIL on this one so let's do that.

$$\left(3x+5\right)\left(3x-5\right)=9x^2-15x+15x-25=9x^2-25$$

In this case the middle terms drop out.

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(b) $\left(2x+6\right)^2$

Now recall that $4^2=\left(4\right)\left(4\right)=16$. Squaring with polynomials works the same way. So in this case we have,

$$\left(2x+6\right)^2=\left(2x+6\right)\left(2x+6\right)=4x^2+12x+12x+36=4x^2+24x+36$$

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(c) $\left(1-7x\right)^2$

This one is nearly identical to the previous part.

$$\left(1-7x\right)^2=\left(1-7x\right)\left(1-7x\right)=1-7x-7x+49x^2=1-14x+49x^2$$

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(d) $4(x+3)^2$

This part is here to remind us that we need to be careful with coefficients. When we've got a coefficient we MUST do the exponentiation first and then multiply the coefficient.

$$4(x+3)^2 = 4(x+3)(x+3) = 4(x^2+6x+9) = 4x^2+24x+36$$

You can only multiply a coefficient through a set of parenthesis if there is an exponent of "1" on the parenthesis. If there is any other exponent then you CAN'T multiply the coefficient through the parenthesis.

Just to illustrate the point.

$$4(x+3)^2 \neq (4x+12)^2 = (4x+12)(4x+12) = 16x^2+96x+144$$

This is clearly not the same as the correct answer so be careful!

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The parts of this example all use one of the following special products.

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Be careful to not make the following mistakes!

$$(a+b)^2 \neq a^2 + b^2$$
$$(a-b)^2 \neq a^2 - b^2$$

These are very common mistakes that students often make when they first start learning how to multiply polynomials.

Factoring Polynomials

Of all the topics covered in this chapter factoring polynomials is probably the most important topic. There are many sections in later chapters where the first step will be to factor a polynomial. So, if you can't factor the polynomial then you won't be able to even start the problem let alone finish it.

Let's start out by talking a little bit about just what factoring is. Factoring is the process by which we go about determining what we multiplied to get the given quantity. We do this all the time with numbers. For instance, here are a variety of ways to factor 12.

$$12 = (2)(6) \qquad 12 = (3)(4) \qquad 12 = (2)(2)(3)$$
$$12 = \left(\frac{1}{2}\right)(24) \qquad 12 = (-2)(-6) \qquad 12 = (-2)(2)(-3)$$

There are many more possible ways to factor 12, but these are representative of many of them.

A common method of factoring numbers is to **completely factor** the number into positive prime factors. A **prime** number is a number whose only positive factors are 1 and itself. For example 2, 3, 5, and 7 are all examples of prime numbers. Examples of numbers that aren't prime are 4, 6, and 12 to pick a few.

If we completely factor a number into positive prime factors there will only be one way of doing it. That is the reason for factoring things in this way. For our example above with 12 the complete factorization is,

$$12 = (2)(2)(3)$$

Factoring polynomials is done in pretty much the same manner. We determine all the terms that were multiplied together to get the given polynomial. We then try to factor each of the terms we found in the first step. This continues until we simply can't factor anymore. When we can't do any more factoring we will say that the polynomial is **completely factored.**

Here are a couple of examples.

$$x^2 - 16 = (x+4)(x-4)$$

This is completely factored since neither of the two factors on the right can be further factored.

Likewise,

$$x^4 - 16 = (x^2+4)(x^2-4)$$

is not completely factored because the second factor can be further factored. Note that the first factor is completely factored however. Here is the complete factorization of this polynomial.

$$x^4 - 16 = (x^2+4)(x+2)(x-2)$$

The purpose of this section is to familiarize ourselves with many of the techniques for factoring polynomials.

Greatest Common Factor

The first method for factoring polynomials will be factoring out the **greatest common factor**. When factoring in general this will also be the first thing that we should try as it will often simplify the problem.

To use this method all that we do is look at all the terms and determine if there is a factor that is in common to all the terms. If there is, we will factor it out of the polynomial. Also note that in this case we are really only using the distributive law in reverse. Remember that the distributive law states that

$$a\left(b+c\right)=ab+ac$$

In factoring out the greatest common denominator we do this in reverse. We notice that each term has an a in it and so we "factor" it out using the distributive law in reverse as follows,

$$ab+ac=a\left(b+c\right)$$

Let's take a look at some examples.

Example 1 Factor out the greatest common factor from each of the following polynomials.

(a) $8x^4-4x^3+10x^2$ [Solution]

(b) $x^3y^2+3x^4y+5x^5y^3$ [Solution]

(c) $3x^6-9x^2+3x$ [Solution]

(d) $9x^2\left(2x+7\right)-12x\left(2x+7\right)$ [Solution]

Solution

(a) $8x^4-4x^3+10x^2$

First we will notice that we can factor a 2 out of every term. Also note that we can factor an x^2 out of every term. Here then is the factoring for this problem.

$$8x^4-4x^3+10x^2=2x^2\left(4x^2-2x+5\right)$$

Note that we can always check our factoring by multiplying the terms back out to make sure we get the original polynomial.

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(b) $x^3y^2+3x^4y+5x^5y^3$

In this case we have both x's and y's in the terms but that doesn't change how the process works. Each term contains and x^3 and a y so we can factor both of those out. Doing this gives,

$$x^3y^2+3x^4y+5x^5y^3=x^3y\left(y+3x+5x^2y^2\right)$$

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(c) $3x^6-9x^2+3x$

In this case we can factor a $3x$ out of every term. Here is the work for this one.

$$3x^6-9x^2+3x=3x\left(x^5-3x+1\right)$$

Notice the "+1" where the $3x$ originally was in the final term, since the final term was the term we factored out we needed to remind ourselves that there was a term there originally. To do this we need the "+1" and notice that it is "+1" instead of "-1" because the term was originally a positive term. If it had been a negative term originally we would have had to use "-1".

One of the more common mistakes with these types of factoring problems is to forget this "1".

Remember that we can always check by multiplying the two back out to make sure we get the original. To check that the "+1" is required, let's drop it and then multiply out to see what we get.

$$3x\left(x^5 - 3x\right) = 3x^6 - 9x^2 \neq 3x^6 - 9x^2 + 3x$$

So, without the "+1" we don't get the original polynomial! Be careful with this. It is easy to get in a hurry and forget to add a "+1" or "-1" as required when factoring out a complete term.

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(d) $9x^2\left(2x+7\right) - 12x\left(2x+7\right)$

This one looks a little odd in comparison to the others. However, it works the same way. There is a $3x$ in each term and there is also a $2x+7$ in each term and so that can also be factored out. Doing the factoring for this problem gives,

$$9x^2\left(2x+7\right) - 12x\left(2x+7\right) = 3x\left(2x+7\right)\left(3x-4\right)$$

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Factoring By Grouping

This is a method that isn't used all that often, but when it can be used it can be somewhat useful. This method is best illustrated with an example or two.

Example 2 Factor by grouping each of the following.

(a) $3x^2 - 2x + 12x - 8$ [Solution]

(b) $x^5 + x - 2x^4 - 2$ [Solution]

(c) $x^5 - 3x^3 - 2x^2 + 6$ [Solution]

Solution

(a) $3x^2 - 2x + 12x - 8$

In this case we *group* the first two terms and the final two terms as shown here,

$$\left(3x^2 - 2x\right) + \left(12x - 8\right)$$

Now, notice that we can factor an x out of the first grouping and a *4* out of the second grouping. Doing this gives,

$$3x^2 - 2x + 12x - 8 = x\left(3x-2\right) + 4\left(3x-2\right)$$

We can now see that we can factor out a common factor of $3x-2$ so let's do that to the final factored form.

$$3x^2 - 2x + 12x - 8 = \left(3x-2\right)\left(x+4\right)$$

And we're done. That's all that there is to factoring by grouping. Note again that this will not always work and sometimes the only way to know if it will work or not is to try it and see what you get.

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(b) $x^5 + x - 2x^4 - 2$

In this case we will do the same initial step, but this time notice that both of the final two terms are negative so we'll factor out a "-" as well when we group them. Doing this gives,

$$\left(x^5 + x\right) - \left(2x^4 + 2\right)$$

Again, we can always distribute the "-" back through the parenthesis to make sure we get the original polynomial.

At this point we can see that we can factor an *x* out of the first term and a 2 out of the second term. This gives,

$$x^5 + x - 2x^4 - 2 = x\left(x^4 + 1\right) - 2\left(x^4 + 1\right)$$

We now have a common factor that we can factor out to complete the problem.

$$x^5 + x - 2x^4 - 2 = \left(x^4 + 1\right)\left(x - 2\right)$$

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(c) $x^5 - 3x^3 - 2x^2 + 6$

This one also has a "-" in front of the third term as we saw in the last part. However, this time the fourth term has a "+" in front of it unlike the last part. We will still factor a "-" out when we group however to make sure that we don't lose track of it. When we factor the "-" out notice that we needed to change the "+" on the fourth term to a "-". Again, you can always check that this was done correctly by multiplying the "-" back through the parenthsis.

$$\left(x^5 - 3x^3\right) - \left(2x^2 - 6\right)$$

Now that we've done a couple of these we won't put the remaining details in and we'll go straight to the final factoring.

$$x^5 - 3x^3 - 2x^2 + 6 = x^3\left(x^2 - 3\right) - 2\left(x^2 - 3\right) = \left(x^2 - 3\right)\left(x^3 - 2\right)$$

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Factoring by grouping can be nice, but it doesn't work all that often. Notice that as we saw in the last two parts of this example if there is a "-" in front of the third term we will often also factor that out of the third and fourth terms when we group them.

Factoring Quadratic Polynomials

First, let's note that quadratic is another term for second degree polynomial. So we know that the largest exponent in a quadratic polynomial will be a 2. In these problems we will be attempting to factor quadratic polynomials into two first degree (hence forth linear) polynomials. Until you become good at these, we usually end up doing these by trial and error although there are a couple of processes that can make them somewhat easier.

Let's take a look at some examples.

Example 3 Factor each of the following polynomials.

(a) $x^2 + 2x - 15$ [Solution]

(b) $x^2 - 10x + 24$ [Solution]

(c) $x^2 + 6x + 9$ [Solution]

(d) $x^2 + 5x + 1$ [Solution]

(e) $3x^2 + 2x - 8$ [Solution]

(f) $5x^2 - 17x + 6$ [Solution]

(g) $4x^2 + 10x - 6$ [Solution]

Solution

(a) $x^2 + 2x - 15$

Okay since the first term is x^2 we know that the factoring must take the form.

$$x^2 + 2x - 15 = (x + \_)(x + \_)$$

We know that it will take this form because when we multiply the two linear terms the first term must be x^2 and the only way to get that to show up is to multiply x by x. Therefore, the first term in each factor must be an x. To finish this we just need to determine the two numbers that need to go in the blank spots.

We can narrow down the possibilities considerably. Upon multiplying the two factors out these two numbers will need to multiply out to get -15. In other words these two numbers must be factors of -15. Here are all the possible ways to factor -15 using only integers.

$$(-1)(15) \qquad (1)(-15) \qquad (-3)(5) \qquad (3)(-5)$$

Now, we can just plug these in one after another and multiply out until we get the correct pair. However, there is another trick that we can use here to help us out. The correct pair of numbers must add to get the coefficient of the x term. So, in this case the third pair of factors will add to "+2" and so that is the pair we are after.

Here is the factored form of the polynomial.

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

Again, we can always check that we got the correct answer my doing a quick multiplication.

Note that the method we used here will only work if the coefficient of the x^2 term is one. If it is anything else this won't work and we really will be back to trial and error to get the correct factoring form.

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(b) $x^2 - 10x + 24$

Let's write down the initial form again,

$$x^2 - 10x + 24 = (x + \_)(x + \_)$$

Now, we need two numbers that multiply to get 24 and add to get -10. It looks like -6 and -4 will do the trick and so the factored form of this polynomial is,

$$x^2 - 10x + 24 = (x - 4)(x - 6)$$

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(c) $x^2 + 6x + 9$

Again, let's start with the initial form,

$$x^2 + 6x + 9 = (x + \_)(x + \_)$$

This time we need two numbers that multiply to get 9 and add to get 6. In this case 3 and 3 will be the correct pair of numbers. Don't forget that the two numbers can be the same number on occasion as they are here.

Here is the factored form for this polynomial.

$$x^2+6x+9=(x+3)(x+3)=(x+3)^2$$

Note as well that we further simplified the factoring to acknowledge that it is a perfect square. You should always do this when it happens.

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(d) x^2+5x+1

Once again, here is the initial form,

$$x^2+5x+1=(x+\_\,)(x+\_\,)$$

Okay, this time we need two numbers that multiply to get 1 and add to get 5. There aren't two integers that will do this and so this quadratic doesn't factor.

This will happen on occasion so don't get excited about it when it does.

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(e) $3x^2+2x-8$

Okay, we no longer have a coefficient of 1 on the x^2 term. However we can still make a guess as to the initial form of the factoring. Since the coefficient of the x^2 term is a 3 and there are only two positive factors of 3 there is really only one possibility for the initial form of the factoring.

$$3x^2+2x-8=(3x+\_\,)(x+\_\,)$$

Since the only way to get a $3x^2$ is to multiply a $3x$ and an x these must be the first two terms. However, finding the numbers for the two blanks will not be as easy as the previous examples. We will need to start off with all the factors of -8.

$$(-1)(8) \qquad (1)(-8) \qquad (-2)(4) \qquad (2)(-4)$$

At this point the only option is to pick a pair plug them in and see what happens when we multiply the terms out. Let's start with the fourth pair. Let's plug the numbers in and see what we get.

$$(3x+2)(x-4)=3x^2-10x-8$$

Well the first and last terms are correct, but then they should be since we've picked numbers to make sure those work out correctly. However, since the middle term isn't correct this isn't the correct factoring of the polynomial.

That doesn't mean that we guessed wrong however. With the previous parts of this example it didn't matter which blank got which number. This time it does. Let's flip the order and see what we get.

$$(3x-4)(x+2)=3x^2+2x-8$$

So, we got it. We did guess correctly the first time we just put them into the wrong spot.

So, in these problems don't forget to check both places for each pair to see if either will work.

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(f) $5x^2-17x+6$

Again the coefficient of the x^2 term has only two positive factors so we've only got one possible initial form.

$$5x^2 - 17x + 6 = (5x + \_)(x + \_)$$

Next we need all the factors of 6. Here they are.

$$(1)(6) \qquad (-1)(-6) \qquad (2)(3) \qquad (-2)(-3)$$

Don't forget the negative factors. They are often the ones that we want. In fact, upon noticing that the coefficient of the x is negative we can be assured that we will need one of the two pairs of negative factors since that will be the only way we will get negative coefficient there. With some trial and error we can get that the factoring of this polynomial is,

$$5x^2 - 17x + 6 = (5x - 2)(x - 3)$$

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(g) $4x^2 + 10x - 6$

In this final step we've got a harder problem here. The coefficient of the x^2 term now has more than one pair of positive factors. This means that the initial form must be one of the following possibilities.

$$4x^2 + 10x - 6 = (4x + \_)(x + \_)$$

$$4x^2 + 10x - 6 = (2x + \_)(2x + \_)$$

To fill in the blanks we will need all the factors of -6. Here they are,

$$(-1)(6) \qquad (1)(-6) \qquad (-2)(3) \qquad (2)(-3)$$

With some trial and error we can find that the correct factoring of this polynomial is,

$$4x^2 + 10x - 6 = (2x - 1)(2x + 6)$$

Note as well that in the trial and error phase we need to make sure and plug each pair into both possible forms and in both possible orderings to correctly determine if it is the correct pair of factors or not.

We can actually go one more step here and factor a 2 out of the second term if we'd like to. This gives,

$$4x^2 + 10x - 6 = 2(2x - 1)(x + 3)$$

This is important because we could also have factored this as,

$$4x^2 + 10x - 6 = (4x - 2)(x + 3)$$

which, on the surface, appears to be different from the first form given above. However, in this case we can factor a 2 out of the first term to get,

$$4x^2 + 10x - 6 = 2(2x - 1)(x + 3)$$

This is exactly what we got the first time and so we really do have the same factored form of this polynomial.

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Special Forms
There are some nice special forms of some polynomials that can make factoring easier for us on occasion. Here are the special forms.

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Let's work some examples with these.

Example 4 Factor each of the following.

(a) $x^2 - 20x + 100$ [Solution]

(b) $25x^2 - 9$ [Solution]

(c) $8x^3 + 1$ [Solution]

Solution

(a) $x^2 - 20x + 100$

In this case we've got three terms and it's a quadratic polynomial. Notice as well that the constant is a perfect square and its square root is 10. Notice as well that 2(10)=20 and this is the coefficient of the x term. So, it looks like we've got the second special form above. The correct factoring of this polynomial is,

$$x^2 - 20x + 100 = (x-10)^2$$

To be honest, it might have been easier to just use the general process for factoring quadratic polynomials in this case rather than checking that it was one of the special forms, but we did need to see one of them worked.

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(b) $25x^2 - 9$

In this case all that we need to notice is that we've got a difference of perfect squares,

$$25x^2 - 9 = (5x)^2 - (3)^2$$

So, this must be the third special form above. Here is the correct factoring for this polynomial.

$$25x^2 - 9 = (5x+3)(5x-3)$$

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(c) $8x^3 + 1$

This problem is the sum of two perfect cubes,

$$8x^3 + 1 = (2x)^3 + (1)^3$$

and so we know that it is the fourth special form from above. Here is the factoring for this polynomial.

$$8x^3 + 1 = (2x+1)(4x^2 - 2x + 1)$$

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Do not make the following factoring mistake!

$$a^2 + b^2 \neq (a+b)^2$$

This just simply isn't true for the vast majority of sums of squares, so be careful not to make this very common mistake. There are rare cases where this can be done, but none of those special cases will be seen here.

Factoring Polynomials with Degree Greater than 2

There is no one method for doing these in general. However, there are some that we can do so let's take a look at a couple of examples.

Example 5 Factor each of the following.

(a) $3x^4 - 3x^3 - 36x^2$ [Solution]

(b) $x^4 - 25$ [Solution]

(c) $x^4 + x^2 - 20$ [Solution]

Solution

(a) $3x^4 - 3x^3 - 36x^2$

In this case let's notice that we can factor out a common factor of $3x^2$ from all the terms so let's do that first.

$$3x^4 - 3x^3 - 36x^2 = 3x^2\left(x^2 - x - 12\right)$$

What is left is a quadratic that we can use the techniques from above to factor. Doing this gives us,

$$3x^4 - 3x^3 - 36x^2 = 3x^2\left(x-4\right)\left(x+3\right)$$

Don't forget that the **FIRST** step to factoring should always be to factor out the greatest common factor. This can only help the process.

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(b) $x^4 - 25$

There is no greatest common factor here. However, notice that this is the difference of two perfect squares.

$$x^4 - 25 = \left(x^2\right)^2 - \left(5\right)^2$$

So, we can use the third special form from above.

$$x^4 - 25 = \left(x^2 + 5\right)\left(x^2 - 5\right)$$

Neither of these can be further factored and so we are done. Note however, that often we will need to do some further factoring at this stage.

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(c) $x^4 + x^2 - 20$

Let's start this off by working a factoring a different polynomial.

$$u^2 + u - 20 = \left(u-4\right)\left(u+5\right)$$

We used a different variable here since we'd already used x's for the original polynomial.

So, why did we work this? Well notice that if we let $u = x^2$ then $u^2 = \left(x^2\right)^2 = x^4$. We can then rewrite the original polynomial in terms of u's as follows,

$$x^4 + x^2 - 20 = u^2 + u - 20$$

and we know how to factor this! So factor the polynomial in u's then back substitute using the fact that we know $u = x^2$.

$$\begin{aligned} x^4 + x^2 - 20 &= u^2 + u - 20 \\ &= (u-4)(u+5) \\ &= \left(x^2-4\right)\left(x^2+5\right) \end{aligned}$$

Finally, notice that the first term will also factor since it is the difference of two perfect squares. The correct factoring of this polynomial is then,

$$x^4 + x^2 - 20 = (x-2)(x+2)\left(x^2+5\right)$$

Note that this converting to u first can be useful on occasion, however once you get used to these this is usually done in our heads.

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We did not do a lot of problems here and we didn't cover all the possibilities. However, we did cover some of the most common techniques that we are liable to run into in the other chapters of this work.